Free particles beyond fermions and bosons

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It is commonly believed that there are only two types of particle exchange statistics in quantum mechanics, fermions and bosons, with the exception of anyons in two dimension [1–4]. In principle, a second exception known as parastatistics, which extends outside of two dimensions, has been considered [5] but was believed to be physically equivalent to fermions and bosons [6, 7]. In this paper we show that nontrivial parastatistics inequivalent to either fermions or bosons can exist in physical systems. These new types of identical particles obey generalized exclusion principles, leading to exotic free-particle thermodynamics distinct from any system of free fermions and bosons. We formulate our theory by developing a second quantization of paraparticles, which naturally includes exactly solvable non-interacting theories, and incorporates physical constraints such as locality. We then construct a family of one-dimensional quantum spin models where free parastatistical particles emerge as quasiparticle excitations. This demonstrates the possibility of a new type of quasiparticle in condensed matter systems, and, more speculatively, the potential for previously unconsidered types of elementary particles.

Introduction It is commonly believed that there are only two types of particle exchange statistics — fermions and bosons. The standard textbook argument for this dichotomy goes as follows. Each multiparticle quantum state is described by a wavefunction $\Psi(x_1, x_2, \ldots, x_n)$, a complex-valued function of particle coordinates in a d dimensional space $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$. The particles are identical, meaning that when we exchange any two of them (say x_1, x_2), the resulting wavefunction $\Psi(x_2, x_1, \ldots, x_n)$ must represent the same physical state, and therefore can change by at most a constant factor

$$\Psi(x_2, x_1, \ldots, x_n) = c\Psi(x_1, x_2, \ldots, x_n). \qquad (1)$$

If we do a second exchange, we have

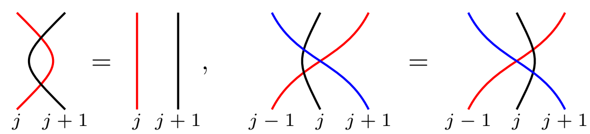
$$\begin{aligned} \Psi(x_1, x_2, \ldots, x_n) &= c\Psi(x_2, x_1, \ldots, x_n) \\ &= c^2\Psi(x_1, x_2, \ldots, x_n), \end{aligned} \qquad (2)$$

leading to $c^2 = 1$, since the wavefunction cannot be constantly zero. This provides exactly two possibilities, bosons ($c = 1$) and fermions ($c = -1$).

Despite being simple and convincing, there are two important exceptions to the fermion/boson dichotomy. The first exception is anyons in two spatial dimension (2D) [1–4, 8], for which c can be any complex phase factor [9]. The second exception that has been considered is parastatistics, which can be consistently defined in any dimension. The way this evades the textbook argument is that the wavefunction can carry extra indices that transform nontrivially during an exchange. Consider an n-particle wavefunction $\Psi^I(x_1, x_2, \ldots, x_n)$, where I is a collection of extra indices corresponding to some internal degrees of freedom inaccessible to physical measurements. Under an exchange between particles j and $j+1$ [10], the wavefunction may undergo a matrix transformation

$$\Psi^I(\{x_j\}_{j=1}^n)|_{x_j \leftrightarrow x_{j+1}} = \sum_J (R_{j,j+1})^I_J \Psi^J(\{x_j\}_{j=1}^n), \qquad (3)$$

for $j = 1, \ldots, n-1$, where the summation is over all possible values of J. Similar to Eq. (1), the matrices $(R_{j,j+1})^I_J$ have to satisfy some algebraic constraints to guarantee consistency of Eq. (3):



(4)

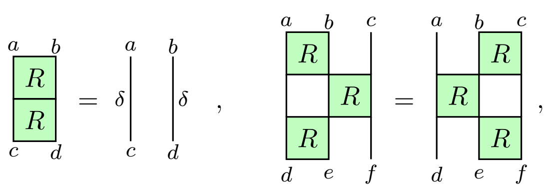
$$R_{j,j+1}^2 = \mathbb{1}, \quad R_{j-1,j}R_{j,j+1}R_{j-1,j} = R_{j,j+1}R_{j-1,j}R_{j,j+1}.$$

The derivation of the first equation is similar to Eq. (2), while the second one follows from the equivalence of two different ways of swapping x_{j-1}, x_j, x_{j+1} to x_{j+1}, x_j, x_{j-1}. These constraints are equivalent to the requirement that $\{R_{j,j+1}\}_{j=1}^{n-1}$ generate a representation of the symmetric group S_n [11]. If this representation is not one-dimensional, we say Eq. (3) defines a type of parastatistical particles, or paraparticles for short.

Parastatistics, and their apparent absence in nature, has been discussed since the dawn of quantum mechanics [12]. The first concrete theory of parastatistics was proposed and investigated by Green in 1952 [5]. This theory was subsequently studied in detail [13–17], and also more generally and rigorously [6, 7, 18] within the framework of algebraic quantum field theory [19, 20]. These works did not rule out the existence of paraparticles in nature, but led to the conclusion that under certain assumptions any theory of paraparticles (in particular, Green's theory) is physically indistinguishable from theories of ordinary fermions and bosons, provided that only local measurements are allowed. This seemingly obviated the need to consider paraparticles, as they give exactly the same physical predictions as theories of ordinary particles.

In this paper we show that nontrivial paraparticles that are inequivalent to any combination of fermions and bosons exist in physical models, in a way compatible with spatial locality and Hermiticity. This poses no contradiction with earlier results, as the construction evades their restrictive assumptions. We demonstrate this by first introducing a second quantization formulation of parastatistics that is distinct from previous constructions, and in this formulation paraparticles display generalized exclusion statistics and free-particle thermodynamics inequivalent to fermions and bosons. Then we show that these paraparticles emerge as quasiparticle excitations in a family of one-dimensional quantum spin models [21], explicitly demonstrating how to avoid the aforementioned no-go theorems [6], allowing nontrivial consequences of parastatistics to be physically observed. Importantly, our construction includes exactly solvable theories of free paraparticles [22], in which paraparticle eigenmodes can be analytically obtained in a similar way as in the solution of free fermions and bosons. Although our spin model realization works only in 1D, our general formulation of paraparticles is valid in any dimension, and we discuss promising directions to realize emergent paraparticles in higher dimensional quantum spin systems, and the potential existence of elementary paraparticles in nature.

Basic formalism We first present our second quantization formulation of parastatistics. This formulation only realizes a subfamily of parastatistics defined by the first quantization approach presented above, but the payoff is that it automatically guarantees the fundamental requirement of spatial locality, which is not ensured by the first quantization formulation [23]. In this formulation, each type of parastatistics is labeled by a four-index tensor R^{ab}_{cd} (where $1 \leq a, b, c, d \leq m$, $m \in \mathbb{Z}$) satisfying the two tensor equations



(5)

where $R^{ab}_{cd} = {}^{a}_{c}\boxed{R}^{b}_{d}$, and throughout this paper we use tensor graphical notation where open indices are identified on both sides of the equation and contracted indices are summed over, and a line segment represents a Kronecker δ function. These two equations are reminiscent of Eq. (4), and we describe their precise relation in the Supplementary Information (SI) [24]. The second equation in Eq. (5) is known in the literature as the constant Yang-Baxter equation (YBE) (whose solutions are called R-matrices), which appears in diverse areas of mathematical physics [25–27]. In Tab. I we present some basic examples of R-matrices, and it can be checked by straightforward computation that they satisfy Eq. (5).

For a given R-matrix, we define the paraparticle creation and annihilation operators $\hat{\psi}^{\pm}_{i,a}$ through the commutation relations (CRs)

Ex.	1	2	3	4
R^{ab}_{cd}	$-\delta_{ad}\delta_{bc}$	$\delta_{ad}\delta_{bc}(-1)^{\delta_{ab}}$	$-\delta_{ac}\delta_{bd}$	$\lambda_{ab}c_{cd} - \delta_{ac}\delta_{bd}$
$z_R(x)$	$(1+x)^m$	$(1+x)^m$	$1+mx$	$1+mx+x^2$

TABLE I. Examples of R-matrices and their single mode partition functions $z_R(x)$, as defined in Eq. (13), where $x = e^{-\beta\epsilon}$. The λ, c in Ex. 4 are $m\times m$ constant matrices satisfying $\lambda c \lambda^T c^T = \mathbb{1}_m$ and $\mathrm{Tr}(\lambda c^T) = 2$ [28].

$$\begin{aligned}
\hat{\psi}^{-}_{i,a}\hat{\psi}^{+}_{j,b} &= \sum_{cd} R^{ac}_{bd}\hat{\psi}^{+}_{j,c}\hat{\psi}^{-}_{i,d} + \delta_{ab}\delta_{ij}, \\
\hat{\psi}^{+}_{i,a}\hat{\psi}^{+}_{j,b} &= \sum_{cd} R^{cd}_{ab}\hat{\psi}^{+}_{j,c}\hat{\psi}^{+}_{i,d}, \\
\hat{\psi}^{-}_{i,a}\hat{\psi}^{-}_{j,b} &= \sum_{cd} R^{ba}_{dc}\hat{\psi}^{-}_{j,c}\hat{\psi}^{-}_{i,d},
\end{aligned} \tag{6}$$

where i, j are mode indices (e.g., position, momentum), and a, b, c, d are internal indices. Notice that $R^{ab}_{cd} = \pm\delta_{ad}\delta_{bc}$ gives back fermions $(-)$ and bosons $(+)$ with an internal degree of freedom.

A crucial structure in our construction is the Lie algebra of contracted bilinear operators defined as

$$\hat{e}_{ij} \equiv \sum_{a=1}^{m} \hat{\psi}^{+}_{i,a}\hat{\psi}^{-}_{j,a}. \tag{7}$$

We show that the space $\{\hat{e}_{ij}\}_{1\leq i,j\leq N}$ is closed under the commutator $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$, and the corresponding Lie algebra is $\mathfrak{gl}_N$. First, using Eq. (6), we have

$$\begin{aligned}
[\hat{e}_{ij}, \hat{\psi}^{+}_{k,b}] &= \delta_{jk}\hat{\psi}^{+}_{i,b}, \\
[\hat{e}_{ij}, \hat{\psi}^{-}_{k,b}] &= -\delta_{ik}\hat{\psi}^{-}_{j,b},
\end{aligned} \tag{8}$$

which leads to

$$[\hat{e}_{ij}, \hat{e}_{kl}] = \delta_{jk}\hat{e}_{il} - \delta_{il}\hat{e}_{kj}. \tag{9}$$

(See Methods for detailed derivation.) Eq. (9) is the CR between the basis elements $\{\hat{e}_{ij}\}_{1\leq i,j\leq N}$ of the $\mathfrak{gl}_N$ Lie algebra, where $\hat{e}_{ij}$ represents the matrix that has 1 in the i-th row and j-th column and zero everywhere else. We will see that this Lie algebra structure enables straightforward construction of theories of paraparticles that obey locality, Hermiticity, and free particle solvability.

In the usual case of fermions, physical observables are composed of even products of fermionic operators. This comes from the physical requirement of locality — local observables supported on disjoint regions (or space-like regions in relativistic quantum field theory) must commute. We define an analog for parastatistics and show they have analogous properties: for each local region of space S, we define a local operator on S to be a sum

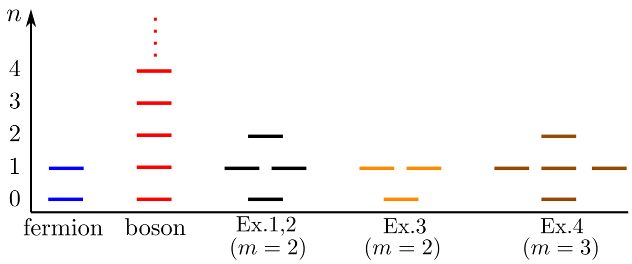


FIG. 1. The generalized exclusion statistics of paraparticles defined by the R-matrices in Ex. 1-4 of Tab. I, and a comparison to ordinary fermions and bosons.

of products of $\hat{e}_{ij}$ where $i,j \in S$. Then, Eq. (9) immediately implies the aforementioned locality condition $[\hat{O}_{S_1}, \hat{O}_{S_2}] = 0$ for $S_1 \cap S_2 = \emptyset$.

Since all physical observables in quantum mechanics are Hermitian, we need to define a Hermitian conjugate $\dagger$ on the state space (which we define later) in order to define local observables. It can be proven [24] that $\dagger$ can be consistently defined on the state space such that $\hat{e}_{ij}^\dagger = \hat{e}_{ji}$, for $1 \leq i,j \leq N$. We define a local observable in the region S to be a local operator $\hat{O}_S$ in S that is Hermitian $\hat{O}_S^\dagger = \hat{O}_S$. For example, $\hat{O}_S = \hat{e}_{ij}\hat{e}_{ji}$ with $i,j \in S$ is a local observable in S. A locally-interacting Hamiltonian $\hat{H}$ is defined to be a sum of local observables $\hat{H} = \sum_S h_S \hat{O}_S$, where $h_S \in \mathbb{R}$ and the summation is over local regions S whose diameters are smaller than some constant cutoff. This definition of local observables and Hamiltonians guarantees microcausality (no signal can travel faster than a finite speed) in both relativistic quantum field theory and non-relativistic lattice quantum systems [29]. It also guarantees the unitarity of the quantum theory, i.e. time evolution $\hat{U} = e^{-i\hat{H}t}$ (setting $\hbar = 1$) generated by a Hamiltonian operator $\hat{H}$ is unitary, and therefore conserves probability.

A particularly important family of physical observables are the particle number operators $\hat{n}_i \equiv \hat{e}_{ii}$. It follows from Eq. (9) that they mutually commute $[\hat{n}_i, \hat{n}_j] = 0$, so they have a complete set of common eigenstates. Meanwhile, Eq. (8) gives $[\hat{n}_i, \hat{\psi}^{\pm}_{j,b}] = \pm\delta_{ij}\hat{\psi}^{\pm}_{j,b}$, meaning that $\hat{\psi}^{+}_{j,b}$ ($\hat{\psi}^{-}_{j,b}$) increases (decreases) the eigenvalue of $\hat{n}_j$ by 1, and does not change the eigenvalue of $\hat{n}_i$ for $j \neq i$. This justifies the terminology creation and annihilation operators, since $\hat{\psi}^{+}_{j,b}$ ($\hat{\psi}^{-}_{j,b}$) creates (annihilates) a particle in the mode j. We also define the total particle number operator $\hat{n} = \sum_{i=1}^N \hat{n}_i$, so we have $[\hat{n}, \hat{\psi}^{\pm}_{j,b}] = \pm\hat{\psi}^{\pm}_{j,b}$. These CRs involving the number operators are the same as for fermions and bosons. However, we will see later that due to the generalized CRs between $\{\hat{\psi}^{\pm}_{i,b}\}$ in Eqs. (6), the spectrum of $\{\hat{n}_i\}$ is different for paraparticles.

The state space To fully define a quantum theory, we need to specify a multiparticle state space and define the action of the creation and annihilation operators $\hat{\psi}^{\pm}_{i,a}$ so that Eq. (6) is satisfied. In the following we state the results without proof, and present the mathematical details in the SI [24]. Analogous to the Fock space of fermions and bosons, there is a vacuum state $|0\rangle$ annihilated by all $\hat{\psi}^{-}_{i,a}$, so the vacuum contains no particles, $\hat{n}|0\rangle = 0$. The state space is spanned by all states of the form $|\psi\rangle = \hat{\psi}^{+}_{i_1,a_1}\hat{\psi}^{+}_{i_2,a_2}\ldots\hat{\psi}^{+}_{i_n,a_n}|0\rangle$. The action of creation operators on this set of states is described straightforwardly by $\hat{\psi}^{+}_{i,a}|\psi\rangle = \hat{\psi}^{+}_{i,a}\hat{\psi}^{+}_{i_1,a_1}\ldots\hat{\psi}^{+}_{i_n,a_n}|0\rangle$. The action of annihilation operators $\hat{\psi}^{-}_{i,a}$ is uniquely determined by the first relation in Eq. (6), which allows us to move $\hat{\psi}^{-}_{i,a}$ all the way to the right until it hits $|0\rangle$ (which it annihilates). A basis for the state space can be constructed as follows. Let $\{\Psi^{\alpha}_{a_1a_2\ldots a_n}\}_{\alpha=1}^{d_n}$ be a complete set of linearly independent solutions to the system of linear equations

$$\sum_{a'_j,a'_{j+1}} R^{a_j a_{j+1}}_{a'_j a'_{j+1}} \Psi_{a_1\ldots a'_j a'_{j+1}\ldots a_n} = \Psi_{a_1\ldots a_j a_{j+1}\ldots a_n} \tag{10}$$

for $j = 1, 2, \ldots, n-1$. Intuitively, Eq. (10) requires that $\Psi_{a_1\ldots a_n}$ is an R-symmetric function, which in the case of fermions or bosons ($R = \pm 1$) reduces to totally symmetric or antisymmetric functions. A basis for the state space is constructed as the set of states of the form [24]

$$|^{\alpha_1}_{n_1}, ^{\alpha_2}_{n_2}, \ldots, ^{\alpha_N}_{n_N}\rangle = \hat{\Psi}^{(1)+}_{n_1,\alpha_1}\hat{\Psi}^{(2)+}_{n_2,\alpha_2}\ldots\hat{\Psi}^{(N)+}_{n_N,\alpha_N}|0\rangle, \tag{11}$$

where the numbers $\{(n_i, \alpha_i)\}_{i=1}^N$ can be chosen independently for different modes (with the only constraint being $1 \leq \alpha_i \leq d_{n_i}$ for each i), and the operator

$$\hat{\Psi}^{(i)+}_{n,\alpha} \equiv \sum_{a_1a_2\ldots a_n} \Psi^{\alpha}_{a_1a_2\ldots a_n}\hat{\psi}^{+}_{i,a_1}\hat{\psi}^{+}_{i,a_2}\ldots\hat{\psi}^{+}_{i,a_n}, \tag{12}$$

creates a multiparticle state in the mode i with occupation number n (eigenvalue of $\hat{n}_i \equiv \hat{e}_{ii}$). In the SI [24], we define a Hermitian inner product on the state space, with respect to which all physical observables are Hermitian, thus completing the construction of a Hilbert space.

The numbers $\{d_n\}_{n\geq 0}$ define the generalized exclusion principle for the paraparticles associated with R. More precisely, a single mode can be occupied by multiple particles, where the n-particle states are d_n-fold degenerate, as illustrated in Fig. 1. This generalizes the Pauli exclusion for fermions (where $d_0 = d_1 = 1$, and $d_n = 0, \forall n \geq 2$), and the Bose-Einstein statistics (where $d_n = 1, \forall n \geq 0$) [30]. Calculation of $\{d_n\}_{n\geq 0}$ for the R-matrices in Exs. 1-4 are given in Methods.

The numbers $\{d_n\}_{n\geq 0}$ allow us to compute the grand canonical partition function for a single mode at temperature T. Suppose that each particle in this mode carries energy ϵ (i.e., the Hamiltonian is $\hat{H} = \epsilon\hat{n}$). Then

$$z_R(e^{-\beta\epsilon}) \equiv \mathrm{Tr}[e^{-\beta\epsilon\hat{n}}] = \sum_{n=0}^{\infty} d_n e^{-n\beta\epsilon}, \tag{13}$$

where $\beta = 1/(k_B T)$, k_B is Boltzmann's constant, and we have absorbed the chemical potential μ into ϵ. The single mode partition functions $z_R(e^{-\beta\epsilon})$ for the R-matrices in Exs. 1-4 are given in Tab. I. Multi-mode partition functions factorize into products of single-mode partition functions exactly as for fermions and bosons.

The single mode partition function $z_R(x)$ (where $x = e^{-\beta\epsilon}$) provides a straightforward demonstration of the non-triviality of the parastatistics for some R-matrices. If the paraparticle system defined by R can be transformed into a system of p flavors of free fermions and q flavors of free bosons, then $z_R(x) = (1+x)^p(1-x)^{-q}$. Therefore the R-matrix given in Ex. 4 must define a non-trivial type of parastatistics for $m \geq 3$, since $z_R(x) = 1 + mx + x^2$ is an irreducible polynomial for $m \geq 3$ and not equal to $(1+x)^p(1-x)^{-q}$ for any integers p, q [31]. When $m = 2$, however, we have $z_R(x) = (1+x)^2$, and it can be shown that the parastatistics defined by this R-matrix is indeed equivalent to two flavors of decoupled fermions. Ex. 3 is similarly non-trivial for $m \geq 2$.

Exact solution of free paraparticles In our second quantization framework, the general bilinear Hamiltonian describing free paraparticles,

$$\hat{H} = \sum_{1\leq i,j\leq N} h_{ij}\hat{e}_{ij} = \sum_{\substack{1\leq i,j\leq N\\ 1\leq a\leq m}} h_{ij}\hat{\psi}^{+}_{i,a}\hat{\psi}^{-}_{j,a}, \tag{14}$$

can be solved analogously to bosons and fermions. We sketch this here; details can be found in Methods. We require $h^*_{ij} = h_{ji}$ so that $\hat{H}^\dagger = \hat{H}$. Using a canonical transformation of $\{\hat{\psi}^{\pm}_{i,a}\}$ that preserves the CRs in Eq. (6), the Hamiltonian can be rewritten as

$$\hat{H} = \sum_{k=1}^{N} \epsilon_k \tilde{n}_k, \tag{15}$$

where $\{\epsilon_k\}_{k=1}^N$ are the eigenvalues of the $N \times N$ coefficient matrix h_{ij}, and $\tilde{n}_k$ is the occupation number operator for the mode k. The operators $\{\tilde{n}_k\}_{k=1}^N$ mutually commute, with common eigenstates defined in Eq. (11) $|^{\alpha_1}_{n_1}, {}^{\alpha_2}_{n_2}, \ldots, {}^{\alpha_N}_{n_N}\rangle$, and the energy eigenvalues are $E = \sum_{k=1}^N \epsilon_k n_k$, where $\{n_k\}_{k=1}^N$ are independent non-negative integers and $1 \leq \alpha_k \leq d_{n_k}$ encodes the single particle exclusion statistics. The average occupation of mode k can be obtained from Eqs. (13) and (15), yielding

$$\langle \tilde{n}_k \rangle_\beta \equiv \frac{\mathrm{Tr}[\tilde{n}_k e^{-\beta\hat{H}}]}{\mathrm{Tr}[e^{-\beta\hat{H}}]} = \frac{z'_R(e^{-\beta\epsilon_k})e^{-\beta\epsilon_k}}{z_R(e^{-\beta\epsilon_k})}. \tag{16}$$

Fig. 2 plots $\langle \tilde{n}_k \rangle_\beta$ as a function of $\beta\epsilon_k$ for the R-matrices in Exs. 3 and 4 (Tab. I) with $m = 5$, showing the distinct finite-temperature thermodynamics of paraparticles compared to ordinary fermions and bosons, characterizing a new type of ideal gas. The thermal average for other physical observables, including correlation functions in and out of equilibrium, can all be calculated exactly.

Discussion Finally, we describe the potential impacts of paraparticles, including routes to observing paraparticles in nature. As part of this, we will describe a family

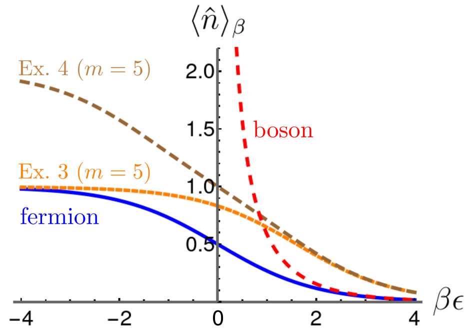


FIG. 2. The thermal expectation value of the single-mode occupation number $\langle \hat{n} \rangle_\beta$: comparison between paraparticles (labeled by the R-matrices in Exs. 3 and 4 of Tab. I with $m = 5$) and ordinary fermions and bosons.

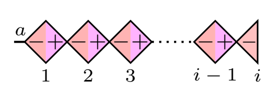
of exactly solvable models with paraparticles as the elementary excitations.

One promising setting for paraparticles is as quasiparticle excitations in condensed matter systems. Significant insight to this direction and a proof-of-principle that such excitations can occur in physical systems is provided by a family of one-dimensional quantum spin systems where free paraparticles emerge as quasiparticle excitations. For each R-matrix, we define a Hamiltonian

$$\hat{H} = \sum_{i,a} J_i(\hat{x}^{+}_{i,a}\hat{y}^{-}_{i+1,a} + \hat{x}^{-}_{i,a}\hat{y}^{+}_{i+1,a}) - \sum_{i,a} \mu_i \hat{y}^{+}_{i,a}\hat{y}^{-}_{i,a}, \tag{17}$$

where $\{\hat{x}^{\pm}_{i,a}, \hat{y}^{\pm}_{i,a}\}_{a=1}^m$ are local spin operators (i.e. operators on different sites commute) acting on the i-th site, whose definition depends on the R-matrix. The index i runs from 1 to N, with N being the system size, and we use open boundary condition $J_N = 0$. The model has a total conserved charge $\hat{n} = \sum_{i,a} \hat{y}^{+}_{i,a}\hat{y}^{-}_{i,a}$ which will be mapped to the paraparticle number operator, and $\hat{x}^{+}_{i,a}, \hat{y}^{+}_{i,a}$ ($\hat{x}^{-}_{i,a}, \hat{y}^{-}_{i,a}$) increase (decrease) $\hat{n}$ by 1. For example, with the R-matrix in Ex. 3, the local Hilbert space $\mathfrak{V}$ is $m+1$-dimensional, with basis states $|0\rangle, \{|1,b\rangle\}_{b=1}^m$, the $\hat{y}^{\pm}_a$ are defined as (omitting the site label) $\hat{y}^{+}_a|0\rangle = |1,a\rangle$, $\hat{y}^{-}_a|1,b\rangle = \delta_{ab}|0\rangle$, $\hat{y}^{-}_a|0\rangle = \hat{y}^{+}_a|1,b\rangle = 0$, and $\hat{x}^{\pm}_a = \hat{y}^{\pm}_a$. This is a simple, nearest-neighbor spin model that is realized in 3-level Rydberg atom or molecule systems [32, 33]. For the definition of $\hat{x}^{\pm}_a$ and $\hat{y}^{\pm}_a$ in general, see SI [24].

This model can be solved using a significant generalization of the Jordan-Wigner transformation (JWT) [34] that we introduce here, in which the products of operators ("strings") are replaced with matrix product operators (MPOs) [35]. Specifically, we introduce operators

$\hat{\psi}^{-}_{ia}$ = 

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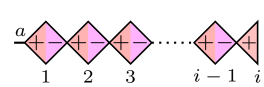
$\hat{\psi}^{+}_{ia}$ = 

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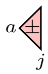
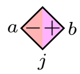
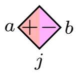
where $\hat{y}^{\pm}_{j,a} \equiv$ , $\hat{S}^{ab}_j \equiv$  $= -[\hat{y}^{+}_{j,a}, \hat{x}^{-}_{j,b}]$, and $\hat{T}^{ab}_j \equiv$  $= +[\hat{y}^{-}_{j,a}, \hat{x}^{+}_{j,b}]$ are local spin operators acting on site j. Both $\hat{\psi}^{\pm}_{ia}$ act non-trivially on sites $1, 2, \ldots, i$ and act as identity on the rest of the chain. For example, with the R-matrix in Ex. 3, $\hat{S}^{ab}$ and $\hat{T}^{ab}$ act as $\hat{S}^{ab}|0\rangle = \hat{T}^{ab}|0\rangle = \delta_{ab}|0\rangle$, $\hat{S}^{ab}|1,c\rangle = -\delta_{bc}|1,a\rangle$, and $\hat{T}^{ab}|1,c\rangle = -\delta_{ac}|1,b\rangle$. The $\hat{\psi}^{\pm}_{i,a}$ constructed in Eq. (18) satisfy the parastatistical CRs in Eq. (6), as we prove in the SI [24] using tensor network manipulations.

The Hamiltonian in Eq. (17) is written in terms of $\{\hat{\psi}^{\pm}_{i,a}\}$ as

$$\hat{H} = \sum_{i,a} J_i(\hat{\psi}^{+}_{i,a}\hat{\psi}^{-}_{i+1,a} + \hat{\psi}^{+}_{i+1,a}\hat{\psi}^{-}_{i,a}) - \sum_i \mu_i \hat{n}_i, \quad (19)$$

so $\hat{\psi}^{\pm}_{i,a_1}$ create/annihilate free emergent paraparticles in the spin model. To gain insight into these excitations, we note that in the special case $m = 1$, $R = -1$, the spin model Eq. (17) is the spin-1/2 XY model, the operators $\hat{\psi}^{\pm}_{i,a}$ are fermionic creation and annihilation operators, and the MPO JWT simplifies to the ordinary JWT.

These results imply a new type of quasiparticle statistics, which can be searched for in condensed matter systems, and a starting point is the exactly solvable quantum spin model defined in Eq. (17). Systems with such excitations may display a wealth of new phenomena. Already non-interacting particles display interesting physics, e.g. the integer quantum Hall effect, topological insulators and superconductors, Bose-Einstein condensation, and Anderson localization, and generalizing these to free paraparticles may lead to new phenomenon qualitatively different from their fermionic or bosonic counterparts. Even more interesting physics occurs with interactions. While interacting paraparticles in general cannot be exactly solved, standard approximation techniques that are based on free particle solvability, such as perturbation theory, several schemes of mean-field theory, renormalization group, and determinantal quantum Monte Carlo simulation [36], can be starightforwardly extended to paraparticles, allowing one to theoretically study these interacting phases of matter.

Our paraparticle theory is well-defined in higher dimensions, so it is natural to consider their potential realization in higher dimensional quantum spin systems. Note that an essential ingredient in finding the exactly solvable 1D realization presented above is the MPO JWT in Eq. (18) [37]. Can we generalize the MPO JWT to higher dimensions to realize paraparticles in higher dimensional spin models? This has been done for fermions: any local Hamiltonian of fermions on a lattice can be mapped to a locally interacting quantum spin model of the same dimension, by an extension of the ordinary JWT [38]. This maps free fermion models to generalized Kitaev's honeycomb models [39–41]. It is therefore reasonable to expect that our MPO JWT (18) can also be extended to higher dimensions, leading to higher dimensional quantum spin models realizing emergent (free) paraparticles, in the same way as the generalized Kitaev models hosting free fermions. One may also gain insights into parastatistics in higher dimensions by constructing Gutzwiller projected wavefunctions [42, 43] with parastatistical partons as variational approximations of certain higher dimensional spin models, using a representation of spin operators in terms of paraparticle operators that generalizes Schwinger's boson and slave fermion representation of SU(2).

In addition to the possibility of emergent parastatistical excitations in interacting quantum matter, a natural, albeit highly speculative, question is to ask if paraparticles may exist as elementary particles in nature. We have seen that our second quantized theory of paraparticles satisfies the fundamental requirements of locality and Hermiticity, and is consistently defined in all dimensions. It is also straightforward to incorporate relativity to get a fully consistent relativistic quantum field theory of elementary paraparticles, in which the canonical quantization of field operators are defined by the R-CRs in Eq. (6). Most fundamental field-theoretical concepts and tools [44] generalize straightforwardly to parastatistics.

In order to consider usefully paraparticles as elementary particles, it is important to consider their superselection rules. Superselection rules arise because the full state space of our second quantized theory is a direct sum of exponentially many subspaces, such that any physical observable has zero matrix element between states of different subspaces. Each subspace is called a superselection sector. An immediate consequence of the superselection rules is that they forbid quantum transitions (by any local unitary evolution) and thermalization between different superselection sectors. If the system is initialized in one sector, it will stay in that sector forever. Therefore the correct thermodynamic description of the system in equilibrium is through the partition function $Z_\pi = \mathrm{Tr}_\pi[e^{-\beta\hat{H}}]$, where Tr_π means summing over all states in a specific sector π, and the result is generally different from that in Fig. 2 obtained by averaging over the whole space (all sectors). In our second quantized theory of paraparticles, it can be shown that [24] the thermal expectation values of all physical observables in a specific sector are the same as some system of ordinary fermions and bosons, meaning that paraparticles in our second quantized theory cannot be distinguished from ordinary particles by local measurements. This is reminiscent of the famous Doplicher-Haag-Roberts (DHR) no-go theorem [6], which states, roughly, that any given superselection sector of a paraparticle system is equivalent to a given fixed particle number sector of a system of fermions and bosons. This problem does not arise for emergent paraparticles in our quantum spin models defined in Eq. (19), which have no such superselection rules, since any two states of the full Hilbert space can be connected by some local spin operators $\{\hat{x}^{\pm}_{i,a}, \hat{y}^{\pm}_{i,a}\}$. Adding an infinitesimal perturbation by such operators will induce thermalization between sectors without perturbing the thermodynamic behavior, allowing the distinct thermodynamics of free paraparticles shown in Fig. 2 to be physically observed. This is similar to how interactions are necessary to thermalize an ideal gas.

The way emergent paraparticles in the quantum spin models in Eq. (19) evade the conclusions of the DHR no-go theorem gives us an important hint on the relevance of parastatistics to elementary particles. Despite being a rigorous mathematical result, the DHR no-go theorem makes several technical assumptions on the physical systems being considered, one of which is the DHR condition [6, 20], which roughly assumes that all excitations are created by local operators. Yet as we see in Eq. (18), the quasiparticles in our spin models are created by nonlocal string operators, thereby rendering inapplicable the DHR theorem in a similar way as the anyonic excitations in Kitaev's toric code model [45, 46], whose creation operators are attached by Z_2 gauge strings. Although our spin models are non-relativistic and limited to 1D, one can potentially introduce other local observables in our general second quantization formalism that are compatible with causality and relativistic covariance but which break the superselection rules, like $\{\hat{x}^{\pm}_{i,a}, \hat{y}^{\pm}_{i,a}\}$ in the spin models. A promising direction is to consider paraparticles coupled to gauge fields, as the DHR theorem does not apply to quantum gauge theories [20], such as Kitaev's toric code model (Z_n gauge theory) and Chern-Simons theories [47], where anyons emerge.

We finally comment that the spin model Hamiltonian in Eq. (17) is non-Hermitian but PT-symmetric if the R-matrix is not Hermitian, as is the case for the R-matrix in Ex. 4 (while other R-matrices in Tab. I are Hermitian, leading to Hermitian spin models). We leave here as an open question if the parastatistics in Ex. 4 is realizable in a Hermitian spin model. Alternatively, since PT-symmetric Hamiltonians can still lead to unitary quantum physics [48], it is interesting to conceive an experimental platform that realizes the non-Hermitian Hamiltonian in Eq. (17) for the R-matrix in Ex. 4.

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[9] For non-Abelian anyons, c is a matrix acting on some internal indices of the wavefunction. This corresponds to our Eq. (3) where $\{R_{j,j+1}\}_{j=1}^{n-1}$ satisfy the second equation in Eq. (4), but not the first one. In 2D, while parastatistics can be considered as a special case of non-Abelian statistics, to the best of our knowledge, this special case has never been studied in the literature of non-Abelian anyons either theoretically or experimentally. Moreover, it is not known that there exists a subfamily of non-Abelian anyons that contains free particles that are not bosons or fermions.
[10] Note that we only need to specify the behavior of the wavefunction under exchange of particles with adjacent labels, since exchange of particles with nonadjacent labels can always be decomposed into a series of adjacent exchanges. For example, under the exchange of particles 1 and 3, the wavefunction should multiply by the matrix $R_{12}R_{23}R_{12}$.
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[21] This is in contrast to Green's paraparticles [5], which have never been proposed as emergent quasiparticles in quantum spin systems. Moreover, while Green's parastatistics is equivalent to a special case of our construction, defined by the R-matrix in Ex. 2 in Tab. I, it can be shown that the corresponding spin model defined in Eq. (17) that realizes this family of parastatistics is equivalent to a collection of decoupled XY models with free fermion excitations.

[22] To our knowledge, this is the first known family of free particle statistics beyond fermions and bosons. A small exception is the free parafermion introduced in Ref. [49] (one should not confuse parafermions with paraparticles, as they are based on a very different generalization of the fermion algebra). Free parafermions can only be defined in 1+1D, and their Hamiltonians are always non-Hermitian with complex energy spectrum. One should also be careful about the use of "paraboson" in the literature — some of them refer to Green's parastatistics [5], while others refer to the q-deformed bosons [50]. The former is physically equivalent to fermions and bosons as mentioned above, and the latter does not contain free particle theories.

[23] It is hard to guarantee locality in the first quantization formulation, and without locality such theories are unable to be realized as elementary particles or as emergent quasiparticle excitations in locally interacting systems. Part of the difficulty comes from the fact that locality puts stringent restrictions on the representation of S_n realized by $\{R_{j,j+1}\}_{j=1}^{n-1}$. For example, a necessary condition for locality is the cluster law introduced in some earlier works [51, 52] (their first quantization formulation of parastatistics is slightly different from ours presented above but closely related, and the cluster law applies to our formulation as well). But the cluster law is only a necessary condition, it is far from clear that it is sufficient (indeed, we suspect it is not).

[24] See Supplementary Information (SI) for the definition of Hermitian inner product on the state space, the rigorous construction of the state space and the action of $\hat{\psi}^{\pm}_{i,a}$, the proof that the generalized JWT maps the spin model to free paraparticles, and a detailed discussion of the superselection structure of our second quantized theory.

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[28] For example, we can take $\lambda = e^{-M}, c = -e^{M}$, where M is an $m \times m$ antisymmetric matrix, $M^T = -M$, with complex entries satisfying $\mathrm{Tr}[e^{-2M}] = -2$ (one can get an explicit solution using a block-diagonal ansatz for M with maximum block size 2).

[29] For the former, the commutativity of local observables at space-like separations rules out faster-than-light travel and communication; for the latter, Ref. [53] proved that as long as all the local Hamiltonian terms have uniformly bounded norms and their algebra has a local structure, then the Lieb-Robinson bound [54] holds, which gives an effective lightcone of causality.

[30] Note that our generalization of Pauli principle is different from that introduced in Ref. [55]. The latter is not compatible with second quantization beyond the simple case of fermions and bosons. In particular, the state counting formula in Eq. (3) of Ref. [55] does not apply to paraparticles introduced in this paper.

[31] We can also show that the average occupation number $\langle\hat{n}\rangle_\beta$ of this type of paraparticles can not be reproduced by fermions and bosons of any energy. Assume the opposite, that $\langle\hat{n}\rangle_\beta$ can be reproduced by a system of p flavors of free fermions and q flavors of free bosons with possibly different energy, i.e. $\langle\hat{n}\rangle_\beta = (mx+2)/(x^2+mx+1) = \sum_{i=1}^{p} x_i/(1+x_i)+\sum_{j=1}^{q} y_j/(1-y_j)$, where $x_i = e^{-\beta\epsilon_i}$ and $y_j = e^{-\beta\epsilon'_j}$. Taking the limit $\beta \to 0$ gives us $p = 2, q = 0$, and the equation simplifies to $1/(1+\lambda^{-1}e^{-\beta\epsilon}) + 1/(1+\lambda e^{-\beta\epsilon}) = 1/(1+e^{-\beta\epsilon_1}) + 1/(1+e^{-\beta\epsilon_2})$, where λ, λ^{-1} are roots of $1+mx+x^2$. As long as $m \geq 3$ and ϵ_1, ϵ_2 are independent of β, this equation cannot be satisfied for all $\beta > 0$.

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[37] Indeed, our logic in Eqs. (17-19) can be reversed: one can insert the MPO JWT in Eq. (18) into the second quantized paraparticle Hamiltonian Eq. (19) to obtain its spin model realization in Eq. (17). This allows us to realize any 1D locally interacting paraparticle theory in 1D local quantum spin models.

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Methods

Derivations for Eqs. (8,9) The commutator between $\hat{e}_{ij}$ and $\hat{\psi}^{\pm}_{k,b}$ is

$$\begin{aligned}[\hat{e}_{ij},\hat{\psi}^{+}_{k,b}] &= \sum_a \left(\hat{\psi}^{+}_{i,a}\hat{\psi}^{-}_{j,a}\hat{\psi}^{+}_{k,b}-\hat{\psi}^{+}_{k,b}\hat{\psi}^{+}_{i,a}\hat{\psi}^{-}_{j,a}\right)\\ &= \sum_a \hat{\psi}^{+}_{i,a}\left(\sum_{c,d}R^{ac}_{bd}\hat{\psi}^{+}_{k,c}\hat{\psi}^{-}_{j,d}+\delta_{jk}\delta_{ab}\right)\\ &\quad -\sum_a \hat{\psi}^{+}_{k,b}\hat{\psi}^{+}_{i,a}\hat{\psi}^{-}_{j,a}\\ &= \sum_{a,c,d}(R^{ac}_{bd}\hat{\psi}^{+}_{i,a}\hat{\psi}^{+}_{k,c})\hat{\psi}^{-}_{j,d}+\delta_{jk}\hat{\psi}^{+}_{i,b}\\ &\quad -\sum_a \hat{\psi}^{+}_{k,b}\hat{\psi}^{+}_{i,a}\hat{\psi}^{-}_{j,a}\\ &= \delta_{jk}\hat{\psi}^{+}_{i,b},\end{aligned}\tag{20}$$

where in the second (third) line we used the first (second) line of Eq. (6). Similarly, we have

$$[\hat{e}_{ij},\hat{\psi}^{-}_{k,b}] = -\delta_{ik}\hat{\psi}^{-}_{j,b}.\tag{21}$$

Now we can compute the commutator

$$\begin{aligned}[\hat{e}_{ij},\hat{e}_{kl}] &= \sum_b[\hat{e}_{ij},\hat{\psi}^{+}_{k,b}]\hat{\psi}^{-}_{l,b}+\sum_b\hat{\psi}^{+}_{k,b}[\hat{e}_{ij},\hat{\psi}^{-}_{l,b}]\\ &= \sum_b\delta_{jk}\hat{\psi}^{+}_{i,b}\hat{\psi}^{-}_{l,b}-\sum_b\hat{\psi}^{+}_{k,b}\delta_{il}\hat{\psi}^{-}_{j,b}\\ &= \delta_{jk}\hat{e}_{il}-\delta_{il}\hat{e}_{kj},\end{aligned}\tag{22}$$

where in the second line we used Eqs. (20) and (21).

Calculation of the exclusion statistics and single mode partition functions We here present the calculation of the numbers $\{d_n\}_{n\geq 0}$ for the R-matrices in Exs. 1-4. To this end, we need to solve Eq. (10) for each R-matrix and for each particle number n. Note that Eq. (10) does not put any restriction on Ψ for $n=0$ and $n=1$, so we have $d_0=1, d_1=m$ for all the four families of R-matrices. The physical meaning of this is clear: we always have one vacuum state $|0\rangle$, and m degenerate single particle states. For the R-matrix in Ex. 1, Eq. (10) sets the requirement that $\Psi_{a_1a_2\ldots a_n}$ is antisymmetric under the exchange of any two neighboring indices, e.g., $\Psi_{a_1a_2\ldots a_n}=-\Psi_{a_2a_1\ldots a_n}$. Therefore for each n, Ψ has $\binom{m}{n}$ independent components, which can be chosen to be $\{\Psi_{a_1a_2\ldots a_n}\mid 1\leq a_1<a_2<\ldots<a_n\leq m\}$, therefore, $d_n=\binom{m}{n}$ for $0\leq n\leq m$ and $d_n=0$ for $n>m$. For the R-matrix in Ex. 2, Eq. (10) still relates an arbitrary component $\Psi_{a_1a_2\ldots a_n}$ to an element in $\{\Psi_{a_1a_2\ldots a_n}\mid 1\leq a_1<a_2<\ldots<a_n\leq m\}$, although potentially with a different sign factor, and we still have $\Psi_{a_1a_2\ldots a_n}=0$ if any two indices are equal. This leads to the same d_n as in Ex. 1. For the R-matrix in Ex. 3, Eq. (10) becomes $\Psi_{a_1a_2\ldots a_n}=-\Psi_{a_1a_2\ldots a_n}$, leading to $\Psi=0$ and therefore $d_n=0$ for any $n\geq 2$. For the R-matrix in Ex. 4, Eq. (10) with $n=2$ gives $\lambda^{ab}\sum_{c,d}c_{cd}\Psi_{cd}=2\Psi_{ab}$, and since $\mathrm{Tr}[\lambda c^T]=2$, this equation has a unique solution $\Psi_{ab}=\lambda_{ab}$ (up to a constant factor), therefore $d_2=1$. Moreover, Eq. (10) with $n=3$ implies $\Psi_{abc}=\lambda_{ab}\phi_c=\phi'_a\lambda_{bc}$ for some vectors ϕ,ϕ', which has no solution since λ is invertible, leading to $d_n=0$ for $n\geq 3$ (the case for $n>3$ is proved by applying this argument to the first 3 indices of $\Psi_{a_1a_2\ldots a_n}$).

The single mode partition function $z_R(x)$ can be calculated directly from the definition in Eq. (13), the results are given in Tab. I. In the mathematics literature $z_R(x)$ (where $x=e^{-\beta\epsilon}$) is called the *Hilbert series* of the R-matrix [56]. There is a very useful identity relating the Hilbert series of the R-matrices R and $-R$ (note that $-R$ also satisfies the YBE in Eq. (5) if R does): $z_R(-x)z_{-R}(x)=1$, which allows us to compute the exclusions statistics $\{d_n\}_{n\geq 0}$ of $-R$ if the exclusions statistics of R is known. For example, for the R-matrix in Ex. 4, we have $z_{-R}(x)=1/(1-mx+x^2)$, from which we obtain $d_0=1, d_1=m, d_2=m^2-1$, and $d_{n+1}=md_n-d_{n-1}$ for $n\geq 1$.

Exact Solution of free paraparticles Here we present details for solving the general bilinear Hamiltonian in Eq. (14). Analogous to usual free bosons and fermions, we consider $U(N)$ transformations of $\{\hat{\psi}^{\pm}_{i,a}\}$:

$$\begin{aligned}\hat{\psi}^{-}_{i,a} &= \sum_{k=1}^{N}U^*_{ki}\tilde{\psi}^{-}_{k,a},\\ \hat{\psi}^{+}_{i,a} &= \sum_{k=1}^{N}U_{ki}\tilde{\psi}^{+}_{k,a},\end{aligned}\tag{23}$$

where U_{ki} is an $N\times N$ unitary matrix, and we use operators with a tilde $\tilde{\psi}^{\pm}_{k,a}$ to denote eigenmode operators. Inserting Eq. (23) into Eq. (6), we see that the operators $\{\tilde{\psi}^{\pm}_{k,a}\}$ satisfy exactly the same CRs as $\{\hat{\psi}^{\pm}_{i,a}\}$. Notice that most of our discussions regarding the second quantization formulation and the state space only assume the CRs in Eq. (6), so the results obtained for $\{\hat{\psi}^{\pm}_{i,a}\}$ (in particular the Lie algebra of bilinear operators and the structure of the state space) must also apply to $\{\tilde{\psi}^{\pm}_{k,a}\}$.

Inserting Eq. (23) into Eq. (14), the Hamiltonian is

$$\hat{H}=\sum_{\substack{1\leq k,p\leq N\\ 1\leq i\leq m}}h'_{kp}\tilde{\psi}^{+}_{k,a}\tilde{\psi}^{-}_{p,a}\equiv\sum_{1\leq k,p\leq N}h'_{kp}\tilde{e}_{kp},\tag{24}$$

where $h'_{kp}=\sum_{1\leq i,j\leq N}U_{ki}h_{ij}U^*_{pj}=[UhU^\dagger]_{kp}$. We can therefore choose the unitary matrix U to diagonalize the Hermitian coefficient matrix h_{ij}, with real eigenvalues $\{\epsilon_k\}_{k=1}^{N}$, and the Hamiltonian becomes diagonal as in Eq. (15).

We now calculate physical observables at temperature T. The partition function is a product of single-mode partition functions in Eq. (13)

$$Z(\beta) \equiv \mathrm{Tr}[e^{-\beta \hat{H}}] = \prod_k z_R(e^{-\beta\epsilon_k}), \tag{25}$$

so the free energy is

$$F(\beta) = -\frac{1}{\beta} \ln Z(\beta) = -\frac{1}{\beta} \sum_k \ln z_R(e^{-\beta\epsilon_k}). \tag{26}$$

The partition function allows us to compute the thermal average of $\hat{n}_k^l$

$$\langle \tilde{n}_k^l \rangle_\beta = \frac{\mathrm{Tr}[\tilde{n}_k^l e^{-\beta\hat{H}}]}{\mathrm{Tr}[e^{-\beta\hat{H}}]} = \left.\frac{(x\partial_x)^l z_R(x)}{z_R(x)}\right|_{x=e^{-\beta\epsilon_k}}. \tag{27}$$

and

$$\langle \tilde{e}_{kp} \rangle_\beta = \delta_{kp} \langle \tilde{n}_k \rangle_\beta \,. \tag{28}$$

The thermal average for physical operators $\hat{e}_{ij}$ are obtained by transforming creation and annihilation operators to the eigenmode basis using Eq. (23), and using the result for $\langle \tilde{e}_{kp} \rangle_\beta$ given in Eq. (28), which yields

$$\langle \hat{e}_{ij} \rangle_\beta = \sum_k U_{ki} U_{kj}^* \langle \tilde{n}_k \rangle_\beta. \tag{29}$$

Observables that are products of $\{\hat{e}_{ij}\}$ (or $\{\tilde{e}_{kp}\}$) can be calculated using the Lie algebra relations in Eq. (9) and the result for $\langle \tilde{e}_{kp} \rangle_\beta$. For example, the thermal expectation value of quadratic products of $\{\hat{e}_{ij}\}$ (e.g. $\hat{e}_{ij}\hat{e}_{ji}$ and $\hat{n}_i\hat{n}_j$) can always be written as linear combinations of $\langle \tilde{e}_{kp}\tilde{e}_{pk} \rangle_\beta$, which can be obtained as follow. First, we have

$$\begin{aligned} \langle \tilde{e}_{kp}\tilde{e}_{pk} \rangle_\beta &= \frac{1}{Z} \mathrm{Tr}[\tilde{e}_{kp}\tilde{e}_{pk} e^{-\beta\hat{H}}] \\ &= \frac{1}{Z} \mathrm{Tr}[\tilde{e}_{pk} e^{-\beta\hat{H}} \tilde{e}_{kp}] \\ &= \langle \tilde{e}_{pk}\tilde{e}_{kp} \rangle_\beta e^{\beta(\epsilon_p - \epsilon_k)}, \end{aligned} \tag{30}$$

where in the last line we used $[\hat{H}, \tilde{e}_{kp}] = (\epsilon_k - \epsilon_p)\tilde{e}_{kp}$. On the other hand, using Eq. (9), we have

$$\begin{aligned} \langle \tilde{e}_{kp}\tilde{e}_{pk} \rangle_\beta - \langle \tilde{e}_{pk}\tilde{e}_{kp} \rangle_\beta &\equiv \langle [\tilde{e}_{kp}, \tilde{e}_{pk}] \rangle_\beta \\ &= \langle \tilde{n}_k \rangle_\beta - \langle \tilde{n}_p \rangle_\beta. \end{aligned} \tag{31}$$

Combining Eqs. (30) and (31), for $p \neq k$, we get

$$\langle \tilde{e}_{kp}\tilde{e}_{pk} \rangle_\beta = \frac{\langle \tilde{n}_k \rangle_\beta - \langle \tilde{n}_p \rangle_\beta}{1 - e^{\beta(\epsilon_k - \epsilon_p)}}. \tag{32}$$

The expression for $p = k$ can be obtained from Eq. (27) by setting $l = 2$. Higher order products of $\{\tilde{e}_{kp}\}$ can be obtained in a similar way, using the structure of the Lie algebra $\mathfrak{gl}_N$, which fulfills a role for paraparticles analogous to Wick's theorem. We can also calculate the unequal time correlators between physical observables, for example

$$\langle [\hat{n}_i(t), \hat{n}_j(0)] \rangle_\beta = \sum_{k,p} U_{ki} U_{pi}^* U_{kj}^* U_{pj} \langle \tilde{n}_k - \tilde{n}_p \rangle_\beta e^{it(\epsilon_k - \epsilon_p)}, \tag{33}$$

which is obtained by expanding $\hat{n}_i(t)$ and $\hat{n}_j(0)$ as linear combinations of $\tilde{e}_{kp}$ and then using Eq. (32).

Supplementary Information for "Free particles beyond fermions and bosons"

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This Supplementary Information fills in several technical details omitted in the main text. In Sec. S1 we define a Hermitian inner product on the state space which guarantees the Hermiticity of Hamiltonians and physical observables. In Sec. S2 we present the detailed construction of the state space and prove the linear independence of the basis states constructed in Eq. (11). In Sec. S3 we provide mathematical details on the 1D spin model defined in Eqs. (17-19).

S1. PROOF OF HERMITICITY

We now prove our claim in the main text that the Hermitian conjugate $\dagger$ can be consistently defined on the states and operators such that $e_{ij}^\dagger = \hat{e}_{ji}$, for $1 \leq i, j \leq N$, which guarantees Hermiticity of Hamiltonians and unitarity of quantum time evolution. To define the Hermitian conjugate $\dagger$ of operators, we need to define a Hermitian inner product $\langle \ldots | \ldots \rangle$ on the state space, and then the Hermitian conjugate of an operator $\hat{O}$ is defined as $\langle \Psi | \hat{O}^\dagger \Phi \rangle \equiv \langle \hat{O} \Psi | \Phi \rangle$, for any states $|\Psi\rangle, |\Phi\rangle$. In the following we first show that such an inner product can be consistently defined on the state space such that the induced Hermitian conjugate $\dagger$ satisfies $e_{ij}^\dagger = \hat{e}_{ji}$, $\forall i, j$, and then give a more explicit definition of this inner product.

To begin, we first notice that, with the CRs in Eq. (9), the set of operators $\{\hat{e}_{ij} + \hat{e}_{ji}, i(\hat{e}_{ij} - \hat{e}_{ji}) | 1 \leq i, j \leq N\}$ spans a closed Lie algebra $\mathfrak{u}_N \cong \mathfrak{su}_N \oplus \mathfrak{u}_1$ (over the field of real numbers $\mathbb{R}$), where the $\mathfrak{u}_1$ part is $n \equiv \sum_{i=1}^N \hat{e}_{ii}$, and the $\mathfrak{su}_N$ part is spanned by $\{\hat{e}_{ij} + \hat{e}_{ji}, i(\hat{e}_{ij} - \hat{e}_{ji}) | 1 \leq i < j \leq N\}$ along with $\{\hat{e}_{ii} - \hat{e}_{i+1,i+1} | 1 \leq i \leq N-1\}$. We now invoke the following theorem whose proof can be found in Ref. [1]

Theorem 1. *For each representation ρ of a compact semisimple real Lie algebra $\mathfrak{g}$ on a finite dimensional $\mathbb{C}$-vector space V, there exists a Hermitian inner product on V such that all $\rho(x) (x \in \mathfrak{g})$ are Hermitian.*

Since $\mathfrak{su}_N$ is a compact semisimple real Lie algebra, Thm. 1 guarantees the existence of a Hermitian inner product such that $\{\hat{e}_{ij} + \hat{e}_{ji}, i(\hat{e}_{ij} - \hat{e}_{ji}) | 1 \leq i < j \leq N\}$ and $\{\hat{e}_{ii} - \hat{e}_{i+1,i+1} | 1 \leq i \leq N-1\}$ are all Hermitian, as long as the state space is finite dimensional. But even if the state space is infinite dimensional, we will see later that the full state space can always be decomposed as a direct sum of finite dimensional irreducible representations (irreps) of $\mathfrak{su}_N$, and Thm. 1 still applies to each irrep. As for the $\mathfrak{u}_1$ part, since n is proportional to the identity operator in each irrep, with the proportionality constant being the total particle number, it follows that n is also Hermitian. Since Thm. 1 implies that $\hat{e}_{ij} + \hat{e}_{ji} = \hat{e}_{ij}^\dagger + \hat{e}_{ji}^\dagger$ and $i(\hat{e}_{ij} - \hat{e}_{ji}) = -i(\hat{e}_{ij}^\dagger - \hat{e}_{ji}^\dagger)$, it follows that $\hat{e}_{ij}^\dagger = \hat{e}_{ji}$, for all $1 \leq i, j \leq N$.

The Hermitian inner product on the state space can be defined more explicitly as follows. The state space constructed in Eqs. (10-12) decomposes into a direct sum of different particle number sectors, and $\hat{n}$ is proportional to identity in each sector. Each particle number sector further decomposes into a direct sum of irreps of $\mathfrak{su}_N$. We set $\langle \Psi | \Phi \rangle = 0$ if $|\Psi\rangle, |\Phi\rangle$ lie in different irreps (i.e., inequivalent irreps or different copies of equivalent irreps) of $\mathfrak{su}_N$. In this way, n is automatically Hermitian (indeed, it is real and diagonal) and the problem reduces to defining $\langle \ldots | \ldots \rangle$ within each irrep of $\mathfrak{su}_N$.

We now show that within each irrep, the inner product between any two states is uniquely determined (up to a multiplicative factor) by the requirement $\hat{e}_{ij}^\dagger = \hat{e}_{ji}$, $\forall i, j$. We show this in the framework of highest weight theory [2]. For every finite-dimensional irrep V of a finite-dimensional semisimple Lie algebra $\mathfrak{g}$, there exists a unique (up to a multiplicative constant) highest weight vector $|\Lambda\rangle$ (a $|\Lambda\rangle$ that is annihilated by all positive root operators $\hat{e}_\alpha |\Lambda\rangle = 0$), and all other weight vectors $|\Lambda'\rangle$ in V can be constructed by applying negative root operators on $|\Lambda\rangle$, i.e. $|\Lambda'\rangle = \prod_\alpha \hat{e}_{-\alpha} |\Lambda\rangle$, where the product is over some ordered set of positive roots. For the case of $\mathfrak{su}_N$, positive root operators are $\{\hat{e}_{ij} | 1 \leq i < j \leq N\}$, negative root operators are $\{\hat{e}_{ji} | 1 \leq i < j \leq N\}$, while $\{\hat{e}_{ii} - \hat{e}_{i+1,i+1} | 1 \leq i \leq N-1\}$ spans the Cartan subalgebra. Without loss of generality we can assume $\langle \Lambda | \Lambda \rangle = 1$. Then for any two weight vectors $|\Lambda_1\rangle, |\Lambda_2\rangle \in V$, their inner product can be calculated as

$$\begin{aligned} \langle \Lambda_1 | \Lambda_2 \rangle &= \prod_{\alpha,\beta} \langle \Lambda | \hat{e}_{-\beta}^\dagger \hat{e}_{-\alpha} | \Lambda \rangle \\ &= \langle \Lambda | \prod_{\alpha,\beta} \hat{e}_\beta \hat{e}_{-\alpha} | \Lambda \rangle, \end{aligned} \tag{S1}$$

and the last line can be calculated by using the CRs between $\hat{e}_\beta$ and $\hat{e}_{-\alpha}$ (to move all the positive root operators $\hat{e}_\beta$ to the right). Notice that there may be several different ways to represent $|\Lambda_{1,2}\rangle$ in the form $\prod_\alpha \hat{e}_{-\alpha} |\Lambda\rangle$, and consequently there are different ways to compute the same inner product $\langle \Lambda_1 | \Lambda_2 \rangle$. Thm. 1 guarantees that all the different ways of computing $\langle \Lambda_1 | \Lambda_2 \rangle$ give the same result.

S2. MATHEMATICAL DETAILS ON STRUCTURE OF STATE SPACE

In this section we present the detailed construction of the state space on a rigorous mathematical level, and prove several claims we made in the main text. Specifically, we will (1) rigorously define the state space and the action of $\hat{\psi}^{\pm}_{i,b}$ on it; (2) prove that the basis states constructed in Eq. (11) are linearly independent [Note the difference between the objectives (1) and (2) and that of Sec. S1: the latter studies the structure of the state space as a representation of the Lie algebra generated by the $\hat{e}_{ij}$s, which tells us the action of $\hat{e}_{ij}$ on the state space. However, results of Sec. S1 do not define the action of $\hat{\psi}^{\pm}_{i,b}$, and they do not tell us how to determine the structure of the full state space (e.g. the exclusion statistics and the partition function) from the R-matrix. The objectives (1) and (2) in this section solve these issues rigorously.]. We will also show the relation between the second and the first quantization formulation.

We begin by introducing some notations. Denote by $\mathcal{X}_{R,N}$ the unital associative algebra over $\mathbb{C}$ generated by $\{\hat{\psi}^{\pm}_{i,b}|1 \le i \le N, 1 \le b \le m\}$ modulo all the relations in Eq. (6). Define $\mathcal{X}^{+}_{R,N}$ as the (unital) subalgebra of $\mathcal{X}_{R,N}$ generated by all the creation operators $\{\hat{\psi}^{+}_{i,b}|1 \le i \le N, 1 \le b \le m\}$, and similarly $\mathcal{X}^{-}_{R,N}$ the (unital) subalgebra of $\mathcal{X}_{R,N}$ generated by all the annihilation operators $\{\hat{\psi}^{-}_{i,b}|1 \le i \le N, 1 \le b \le m\}$.

An important observation is that the algebra $\mathcal{X}_{R,N}$ can be obtained from $\mathcal{X}_{R,1}$ as

$$\mathcal{X}_{R,N} \cong \mathcal{X}_{\Pi\boxtimes R,1}, \tag{S2}$$

where $\Pi \boxtimes R$ is the direct product R-matrix defined as

$$(\Pi \boxtimes R)^{AB}_{CD} \equiv \Pi^{ij}_{kl} R^{ab}_{cd}, \tag{S3}$$

where we group the spatial index $i = 1, 2, \ldots, N$ and the internal index a in $\hat{\psi}^{\pm}_{i,a}$ into a single collective index: $A = (i, a), B = (j, b), C = (k, c)$ and $D = (l, d)$. Π acts on the spatial part defined as $\Pi^{ij}_{kl} = \delta_{il}\delta_{jk}$, and R acts on the internal part. It is straightforward to check that $\Pi\boxtimes R$ constructed this way also satisfies the YBE Eq. (5), and Eq. (S2) can be checked by comparing the defining CRs of both sides. For this reason, in Secs. S2 A and S2 B we focus on the algebra $\mathcal{X}_R \equiv \mathcal{X}_{R,1}$, but keep in mind that any claim we make on $\mathcal{X}_R$ applies equally well to $\mathcal{X}_{R,N}$ by using the product R-matrix $\Pi \boxtimes R$. We will omit the mode labels i, j and simply write $\hat{\psi}^{\pm}_a$ when there is no confusion.

A. Existence and uniqueness of vacuum state from physical requirements

For the theory to make physical sense, the spectrum of the total particle number operator $\hat{n}$ should be bounded from below. This means that there exists at least one state $|n_{\min}\rangle$ with the smallest eigenvalue $n_{\min}$ of $\hat{n}$. Since $\hat{\psi}^{-}_a$ decreases the eigenvalue of $\hat{n}$ by 1, the minimality of $n_{\min}$ requires that $\hat{\psi}^{-}_a|n_{\min}\rangle = 0, \forall a$, since otherwise $\hat{\psi}^{-}_a|n_{\min}\rangle$ would be an eigenstate of $\hat{n}$ with eigenvalue $n_{\min} - 1 < n_{\min}$. Therefore $\hat{n}|n_{\min}\rangle = \sum_a \hat{\psi}^{+}_a\hat{\psi}^{-}_a|n_{\min}\rangle = 0$, i.e., $n_{\min} = 0$. We call this state the vacuum state, denoted by $|0\rangle$.

It can be proven that in an irrep V of $\mathcal{X}_R$, the vacuum state must be unique. Here is a sketch of the proof by contradiction: assume there exists two linearly independent vacuum states, say $|0\rangle, |0'\rangle \in V$. Then $V_0 = \mathcal{X}^{+}_R|0\rangle$ would be invariant under the action of $\mathcal{X}_R$. To prove this, it is enough to show that V_0 is invariant under all the generators $\hat{\psi}^{\pm}_a$ of $\mathcal{X}_R$: $\hat{\psi}^{+}_a$ leaves V_0 invariant since $\hat{\psi}^{+}_a\mathcal{X}^{+}_R \subseteq \mathcal{X}^{+}_R$, while $\hat{\psi}^{-}_a\mathcal{X}^{+}_R \subseteq \mathcal{X}^{+}_R\hat{\psi}^{-}_a + \mathcal{X}^{+}_R$ according to the first relation in Eq. (6), so $\hat{\psi}^{-}_a V_0 \subseteq V_0$. Therefore, V_0 is a subrepresentation of V. Furthermore, $|0'\rangle \notin V_0$ since the only state in V_0 annihilated by $\hat{n}$ is $|0\rangle$. Therefore, V_0 is a proper subrepresentation of V, contradicting the irreducibility of V.

B. The state space generated by $|0\rangle$ and $\{\hat{\psi}^{+}_a\}$

The algebra $\mathcal{X}_R$ is the special case of the quantum Weyl algebras (QWAs) $A_m(R)$ studied in Ref. [3] with $q = 1$, by identifying x_a with $\hat{\psi}^{+}_a$ and ∂_a with $\hat{\psi}^{-}_a$, and Thm. 1.5 in Ref. [3] provides the rigorous mathematical foundation for the construction of state space:

Theorem 2. *(Thm. 1.5 in Ref. [3]) There is a vector space isomorphism $\mathbb{C}_R\langle\hat{\psi}^{+}_a\rangle \otimes \mathbb{C}_R\langle\hat{\psi}^{-}_a\rangle \cong \mathcal{X}_R$, where $\mathbb{C}_R\langle\hat{\psi}^{+}_a\rangle$ is the unital associative algebra generated by $\{\hat{\psi}^{+}_b|1 \le b \le m\}$, subject to the second relations in Eq. (6), and similarly $\mathbb{C}_R\langle\hat{\psi}^{-}_a\rangle$ is the unital associative algebra generated by $\{\hat{\psi}^{-}_b|1 \le b \le m\}$, subject to the third relations in Eq. (6).*

This theorem extends the simpler fact that, as a vector space, $\mathcal{X}_R$ is spanned by $\mathcal{X}^{+}_R\otimes\mathcal{X}^{-}_R$, since for any monomial of $\hat{\psi}^{+}_1, \ldots, \hat{\psi}^{+}_m, \hat{\psi}^{-}_1, \ldots, \hat{\psi}^{-}_m$ in $\mathcal{X}_R$ (e.g. $\hat{\psi}^{-}_a\hat{\psi}^{+}_b\hat{\psi}^{-}_c\hat{\psi}^{+}_d$), one can always use the first relation in Eq. (6) to "normal order" all $\hat{\psi}^{+}_a$s to the left and $\hat{\psi}^{-}_a$s to the right, leading to a sum of terms, each with at most m creation and m annihilation operators. The non-trivial aspect of this theorem is that $\mathcal{X}^{+}_R \cong \mathbb{C}_R\langle\hat{\psi}^{+}_a\rangle$, and $\mathcal{X}^{-}_R \cong \mathbb{C}_R\langle\hat{\psi}^{-}_a\rangle$, i.e. the relations in the first and third lines of Eq. (6) do not imply any additional relations on the $\hat{\psi}^{+}_a$s other than the second line in Eq. (6). See Ref. [3] for a detailed proof.

We now construct the state space as the representation space of $\mathcal{X}_R$, defined as the canonical left $\mathcal{X}_R$-module $\mathfrak{V} = \mathcal{X}_R/[\sum_a \mathcal{X}_R\hat{\psi}^{-}_a]$ (i.e. the left $\mathcal{X}_R$-module generated by a vacuum state $|0\rangle$ satisfying the relation $\hat{\psi}^{-}_a|0\rangle = 0$ for all a). Then Thm. 2 immediately implies that (see the comment at the end of Sec. 1 in Ref. [3]), as a vector space, $\mathfrak{V} = \mathcal{X}_R|0\rangle = \mathcal{X}_R^+|0\rangle \cong \mathbb{C}_R\langle\hat{\psi}_a^+\rangle$. Furthermore, Thm. 3.2 of Ref. [3] proves that the representation $\mathfrak{V}$ of $\mathcal{X}_R$ is irreducible, and our discussion in Sec. S2 A implies that it is the only irrep of $\mathcal{X}_R$ with the spectrum of n bounded from below.

In the following we find a basis for the state space $\mathfrak{V} \equiv \mathcal{X}_R^+|0\rangle$. We use the eigenvalues of the particle number operator $\hat{n}$ to decompose $\mathfrak{V}$ into a direct sum of eigenspaces of $\hat{n}$: $\mathfrak{V} = \bigoplus_{n\geq 0}\mathfrak{V}_n$. Each subspace $\mathfrak{V}_n$ is spanned by states with fixed particle number

$$\mathfrak{V}_n = \mathrm{span}\{\hat{\psi}_{a_1}^+\hat{\psi}_{a_2}^+\ldots\hat{\psi}_{a_n}^+|0\rangle | 1\leq a_j\leq m, j=1,2,\ldots,n\}. \tag{S4}$$

Notice, however, due to the CR in Eq. (6), the states defined in the RHS of Eq. (S4) are linearly dependent. For example, the state $\hat{\psi}_a^+\hat{\psi}_b^+|0\rangle$ is the same as $\sum_{c,d} R_{ab}^{cd}\hat{\psi}_c^+\hat{\psi}_d^+|0\rangle$. A linearly independent basis for $\mathfrak{V}_n$ is established by the following theorem:

Theorem 3. *The states $\{|n,\alpha\rangle\}_{\alpha=1}^{d_n}$ defined by Eqs. (10-12) (for the case $N=1$) form a complete, linearly independent basis for $\mathfrak{V}_n$.*

We now sketch the proof of Thm. 3. Following our discussion in the previous paragraph, the n-particle space $\mathfrak{V}_n$ of a single mode [defined in Eq. (S4)] can be identified with $\mathbb{C}_R^{(n)}\langle\hat{\psi}_a^+\rangle$, the subspace of $\mathbb{C}_R\langle\hat{\psi}_a^+\rangle$ spanned by all degree n monomials in $\hat{\psi}_a^+$s. So it remains to be proven that $\mathbb{C}_R^{(n)}\langle\hat{\psi}_a^+\rangle$ is isomorphic (as a vector space) to the space of solutions $\Psi_{a_1\ldots a_n}$ to Eq. (10). For convenience, we define a product vector space $\mathfrak{A} \equiv \mathfrak{a}^{\otimes n}$, where $\mathfrak{a}$ is an m-dimensional vector space with basis $\{v_1, v_2, \ldots, v_m\}$. The tensor R_{cd}^{ab} defines a linear map R in the product space $\mathfrak{a}\otimes\mathfrak{a}$ as $R(v_c\otimes v_d) = \sum_{ab} R_{cd}^{ab} v_a\otimes v_b$, and this action is extended to $\mathfrak{a}^{\otimes n}$ as

$$R_{j,j+1} = \underset{(1)}{\mathbb{1}}\otimes\ldots\otimes\underset{(j-1)}{\mathbb{1}}\otimes\underset{(j,j+1)}{R}\otimes\underset{(j+2)}{\mathbb{1}}\otimes\ldots\otimes\underset{(n)}{\mathbb{1}}. \tag{S5}$$

Furthermore, we can associate a tensor $\Psi_{a_1\ldots a_n}$ to a vector in $\mathfrak{a}^{\otimes n}$ through $\Psi = \sum_{a_1\ldots a_n}\Psi_{a_1\ldots a_n} v_{a_1}\otimes v_{a_2}\otimes\ldots\otimes v_{a_n}$. Then Eq. (10) is equivalent to

$$R_{j,j+1}\Psi = \Psi \text{ (in } \mathfrak{a}^{\otimes n}),\ j = 1,2,\ldots,n-1. \tag{S6}$$

In short, we need to prove that $\mathbb{C}_R^{(n)}\langle\hat{\psi}_a^+\rangle$ is isomorphic (as a vector space) to the common eigenspace (with eigenvalue $+1$) of all $R_{j,j+1}$. We have, as a vector space,

$$\mathbb{C}_R^{(n)}\langle\hat{\psi}_a^+\rangle \cong \frac{\mathfrak{a}^{\otimes n}}{\bigoplus_{j=1}^{n-1}(\mathbb{1}-R_{j,j+1})\mathfrak{a}^{\otimes n}}, \tag{S7}$$

since $\mathbb{C}_R\langle\hat{\psi}_a^+\rangle$ is, by definition, isomorphic to the quotient of the tensor algebra $T(\mathfrak{a})$ over the quadratic relations $R(\mathfrak{a}\otimes\mathfrak{a}) = (\mathfrak{a}\otimes\mathfrak{a})$ [4], where $\mathfrak{a}$ is an m-dimensional vector space. We now prove the following lemma:

Lemma 4. *Let $R_1, R_2, \ldots, R_k$ be Hermitian matrices satisfying $R_j^2 = \mathbb{1}, j = 1,2,\ldots,k$, acting on a vector space V. Then*

$$\mathrm{span}\{|\psi\rangle\in V \mid R_j|\psi\rangle = |\psi\rangle, 1\leq j\leq k\} \cong \frac{V}{\bigoplus_{j=1}^{k}(\mathbb{1}-R_j)V}. \tag{S8}$$

Proof. For a Hilbert space $\mathcal{H}$, and a subspace $\mathcal{H}_1\subseteq\mathcal{H}$, the quotient space $\mathcal{H}/\mathcal{H}_1$ is isomorphic to the orthogonal complement $\mathcal{H}_1^\perp$. Therefore we have

$$\begin{aligned}\frac{V}{\bigoplus_{j=1}^{k}(\mathbb{1}-R_j)V} &= [\bigoplus_{j=1}^{k}(\mathbb{1}-R_j)V]^\perp \\ &= \bigcap_{j=1}^{k}[(\mathbb{1}-R_j)V]^\perp \\ &= \bigcap_{j=1}^{k}[(\mathbb{1}+R_j)V],\end{aligned} \tag{S9}$$

where in the second line we used $(V_1\oplus V_2)^\perp = V_1^\perp\cap V_2^\perp$, and in the third line we used the fact that R_j is Hermitian and all its eigenvalues are ±1. Note that $(\mathbb{1}+R_j)/2$ is the projector to the eigenspace of R_j with eigenvalue $+1$, therefore, the last line of Eq. (S9) is the same as the LHS of Eq. (S8). □

Although the R-matrices in our models are not always Hermitian, there always exists a Hermitian inner product on the space $\mathfrak{A} = \mathfrak{a}^{\otimes n}$ with respect to which the matrices $\{R_{j,j+1}\}_{j=1}^{n-1}$ are all Hermitian. This is because any finite-dimensional representation of a finite group is isomorphic to a unitary representation (Theorem 4.6.2 in Ref. [5]), and in our case the matrices $\{R_{j,j+1}\}_{j=1}^{n-1}$ generate the finite group S_n (notice that if $R_{j,j+1}$ is unitary then it is Hermitian since $R_{j,j+1}^2 = \mathbb{1}$). Therefore, Lemma 4 still applies, implying that the RHS of Eq. (S7) is isomorphic to the common eigenspaces of $\{R_{j,j+1}\}_{j=1}^{n-1}$ defined by Eq. (10). This concludes the proof of Thm. 3.

C. Many particle state space

We now prove that the states defined in Eqs. (10-12) form a linearly independent basis for the many particle state space $\mathcal{X}_{R,N}^+|0\rangle$ for any positive integer N. We need the following lemma:

Lemma 5. *There is a vector space isomorphism $\mathcal{X}_{R,N}\cong\mathcal{X}_R^{\otimes N}$. In particular, $\mathcal{X}_R$ is isomorphic to the subalgebra of $\mathcal{X}_{R,N}$ generated by $\{\hat{\psi}_{i,a}^\pm | 1\leq a\leq m\}$, for any $a\in\{1,2,\ldots,N\}$.*

Proof. By Thm. 2, we have (as vector spaces) $\mathcal{X}_R\cong\mathbb{C}_R\langle\hat{\psi}_a^+\rangle\otimes\mathbb{C}_R\langle\hat{\psi}_a^-\rangle$, and $\mathcal{X}_{R,N}\cong\mathbb{C}_{\Pi\boxtimes R}\langle\hat{\psi}_a^+\rangle\otimes\mathbb{C}_{\Pi\boxtimes R}\langle\hat{\psi}_a^-\rangle$, so we only need to prove that (as a vector space) $\mathbb{C}_{\Pi\boxtimes R}\langle\hat{\psi}_a^+\rangle\cong\mathbb{C}_R\langle\hat{\psi}_a^+\rangle^{\otimes N}$. This can be proven by induction on N, where the induction step $N\to N+1$ can be proven in a similar way as Thm. 2. Alternatively, it is straightforward to show that $h_{\Pi\boxtimes R}(x) = h_R(x)^N$, and since for every R-matrix, $\dim \mathbb{C}_R\langle\hat{\psi}_a^+\rangle = h_R(1)$, we have $\dim \mathbb{C}_{\Pi\boxtimes R}\langle\hat{\psi}_a^+\rangle = h_R(1)^N = \dim \mathbb{C}_R\langle\hat{\psi}_a^+\rangle^{\otimes N}$, so as a vector space, $\mathbb{C}_{\Pi\boxtimes R}\langle\hat{\psi}_a^+\rangle \cong \mathbb{C}_R\langle\hat{\psi}_a^+\rangle^{\otimes N}$. □

While Lemma 5 seems natural, the non-trivial part is that the CRs (6) involving any other modes $\hat{\psi}_{j,a}^{\pm}$ (with $j \neq i$) do not give rise to any additional algebraic relations on $\{\hat{\psi}_{i,a}^{\pm}|1 \leq a \leq m\}$. This is a rigorous justification that different modes are mutually independent. Lemma 5 along with Thm. 3 immediately imply that the states defined in Eqs. (10-12) form a linearly independent basis for $\mathcal{X}_{R,N}^+|0\rangle$.

D. The relation between the second and the first quantization formulation

We now show the relation between the second quantized formulation of parastatistics and the first quantized wavefunction formulation. To this end we first show the relation between the R-matrix R_{cd}^{ab} and the coefficients $(R_{j,j+1})_J^I$ appearing in Eq. (3). Let the index I (and similarly for J) be a collection of n auxiliary indices $I = (a_1, a_2, \ldots, a_n)$ labeling the basis states of a product vector space $\mathfrak{A} \equiv \mathfrak{a}^{\otimes n}$ (the internal space of wavefunctions), where the basis of $\mathfrak{a}$ is $\{v_1, v_2, \ldots, v_m\}$. Now let $(R_{j,j+1})_J^I$ be the matrix element of the linear mapping defined in Eq. (S5). With this choice of $R_{j,j+1}$, Eq. (3) becomes (take $n = 3$ and $j = 1$ for example)

$$\Psi^{a_1a_2a_3}(x_2, x_1, x_3) = \sum_{b_1,b_2} R_{b_1b_2}^{a_1a_2}\Psi^{b_1b_2a_3}(x_1, x_2, x_3). \tag{S10}$$

Then all the relations in Eq. (4) reduce to Eq. (5). An isomorphism between the space of n-particle wavefunctions in the first quantization formulation and the subspace of n-particle states in the second quantization formulation is defined as follows: each n-particle wavefunction $\Psi^I(x_1, \ldots, x_n)$ satisfying Eq. (3) [with the R-matrix in Eq. (S5)] corresponds to the n-particle state

$$|\Psi\rangle = \sum_{I,x_1,\ldots,x_n} \Psi^I(x_1, \ldots, x_n)\hat{\psi}_{x_1,a_1}^+ \ldots \hat{\psi}_{x_n,a_n}^+|0\rangle, \tag{S11}$$

That Eq. (S11) indeed defines an isomorphism between the two vector spaces can be seen as follows. Note that Eq. (10) and Eq. (11) (for the case $N = 1$) with the R-matrix $\Pi\boxtimes R$ is the same as Eq. (3) and Eq. (S11) with the R-matrix R, respectively. Then Thm. 3 applied to the algebra $\mathcal{X}_{\Pi\boxtimes R} \cong \mathcal{X}_{R,N}$ shows that a linearly independent basis for the space of n-particle wavefunctions satisfying Eq. (3) correspond to a linearly independent basis for the n-particle subspace $\mathfrak{V}_n$ of the Fock space $\mathcal{X}_{\Pi\boxtimes R}^+|0\rangle \cong \mathcal{X}_{R,N}^+|0\rangle$ via the relation Eq. (S11), thereby establishing the isomorphism.

S3. DETAILS ON THE 1D SPIN MODEL AND THE MPO JWT

In this section we provide mathematical details on the 1D spin model defined in Eqs. (17-19).

A. Model definition for an arbitrary R-matrix

We first define the local spin operators $\{\hat{x}_{i,a}^{\pm}, \hat{y}_{i,a}^{\pm}\}_{a=1}^m$ that appears in the Hamiltonian in Eq. (17), for any given R-matrix. They are constructed to satisfy the following algebraic relations (we omit the site label i since they all act locally on the same site)

$$\begin{aligned}
\hat{y}_a^-\hat{y}_b^+ &= \sum_{c,d} R_{bd}^{ac}\hat{y}_c^+\hat{y}_d^- + \delta_{ab} \\
\hat{y}_a^+\hat{y}_b^+ &= \sum_{c,d} R_{ab}^{cd}\hat{y}_c^+\hat{y}_d^+ \\
\hat{y}_a^-\hat{y}_b^- &= \sum_{c,d} R_{dc}^{ba}\hat{y}_c^-\hat{y}_d^- \\
\hat{x}_a^-\hat{x}_b^+ &= \sum_{c,d} R_{db}^{ca}\hat{x}_c^+\hat{x}_d^- + \delta_{ab} \\
\hat{x}_a^+\hat{x}_b^+ &= \sum_{c,d} R_{ba}^{dc}\hat{x}_c^+\hat{x}_d^+ \\
\hat{x}_a^-\hat{x}_b^- &= \sum_{c,d} R_{cd}^{ab}\hat{x}_c^-\hat{x}_d^- \\
[\hat{x}_a^+, \hat{y}_b^+] &= [\hat{x}_a^-, \hat{y}_b^-] = 0.
\end{aligned} \tag{S12}$$

While these CRs superficially resemble the CRs between paraparticle operators in Eq. (6), the difference is that the spin operators here are strictly local in that they commute on different sites, and therefore are in principle realizable, while the paraparticle operators are generally non-local operators. These CRs are shown graphically in Fig. S2.

We now define a local Hilbert space and a matrix representation of these local spin operators. Notice that the first three CRs in Eq. (S12) are exactly the same as in Eq. (6) for $N = 1$, therefore we can take $\mathfrak{V} = \mathcal{X}_R^+|0\rangle \cong \mathbb{C}_R\langle\hat{\psi}_a^+\rangle$ to be the local Hilbert space of the spin model [6], where the action of $\{\hat{y}_a^{\pm}\}_{a=1}^m$ is defined by $\hat{y}_a^{\pm}|\Psi\rangle = \hat{\psi}_a^{\pm}|\Psi\rangle$, $\forall|\Psi\rangle \in \mathfrak{V}$. The action of $\{\hat{x}_a^+\}_{a=1}^m$ on $\mathfrak{V}$ is defined by $\hat{x}_a^+(\hat{\psi}_{b_1}^+\hat{\psi}_{b_2}^+ \ldots \hat{\psi}_{b_n}^+|0\rangle) = \hat{\psi}_{b_1}^+\hat{\psi}_{b_2}^+ \ldots \hat{\psi}_{b_n}^+\hat{\psi}_a^+|0\rangle \in \mathcal{X}_R^+|0\rangle$, for any $\hat{\psi}_{b_1}^+\hat{\psi}_{b_2}^+ \ldots \hat{\psi}_{b_n}^+|0\rangle \in \mathcal{X}_R^+|0\rangle$. The action of $\{\hat{x}_a^-\}_{a=1}^m$ on $\mathfrak{V}$ is defined by $\hat{x}_a^-(\hat{\psi}_{b_1}^+\hat{\psi}_{b_2}^+ \ldots \hat{\psi}_{b_n}^+|0\rangle) = \hat{x}_a^-\hat{x}_{b_n}^+\hat{x}_{b_{n-1}}^+ \ldots \hat{x}_{b_1}^+|0\rangle$, where it is understood that the RHS is simplified by moving $\hat{x}_a^-$ all the way to the right using the fourth relation in Eq. (S12); for example, $\hat{x}_a^-(\hat{\psi}_b^+\hat{\psi}_c^+|0\rangle) = \hat{x}_a^-\hat{x}_c^+\hat{x}_b^+|0\rangle = (\delta_{ac}\hat{x}_b^+ + \sum_d R_{bc}^{da}\hat{x}_d^+)|0\rangle = (\delta_{ac}\hat{\psi}_b^+ + \sum_d R_{bc}^{da}\hat{\psi}_d^+)|0\rangle \in \mathcal{X}_R^+|0\rangle$. It is straightforward to check that the action of $\{\hat{x}_a^{\pm}, \hat{y}_a^{\pm}\}_{a=1}^m$ defined this way satisfy all the relations in Eq. (S12).

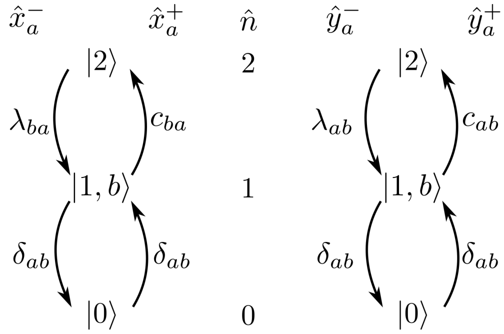

FIG. S1. The action of the operators $\hat{x}_a^\pm, \hat{y}_a^\pm$ and $\hat{n}$ on the basis states $|0\rangle$, $\{|1,b\rangle\}_{b=1}^m$, $|2\rangle$, for the spin model corresponding to the R-matrix in Ex. 4.

Furthermore, we note that this definition implies the additional CRs

$$\begin{aligned}\sum_a \hat{x}_a^+ \hat{x}_a^- \ &= \ \sum_a \hat{y}_a^+ \hat{y}_a^- \equiv \hat{n}, \\ [\hat{n}, \hat{x}_a^\pm] \ &= \ \pm \hat{x}_a^\pm, \quad [\hat{n}, \hat{y}_a^\pm] = \pm \hat{y}_a^\pm. \end{aligned} \tag{S13}$$

The Hilbert space of the whole system with N sites in total is $\mathfrak{V}^{\otimes N}$, and $\{\hat{x}_{i,a}^\pm, \hat{y}_{i,a}^\pm\}_{a=1}^m$ act locally on the i-th factor space as described above.

We can give the matrix elements of $\{\hat{x}_a^\pm, \hat{y}_a^\pm\}_{a=1}^m$ more explicitly for the R-matrices given in Tab. I. One finds that for the R-matrices in Ex. 1 and Ex. 2, the corresponding spin models are simply m decoupled chains of 1D XY models (up to an on-site unitary transformation). The spin model for the R-matrix in Ex. 3 has been presented in the main text. For the R-matrix in Ex. 4, the local Hilbert space $\mathfrak{V}$ is $m+2$-dimensional, with basis states $|0\rangle, \{|1,b\rangle\}_{b=1}^m, |2\rangle$; the non-zero matrix elements of $\hat{y}_a^\pm$ are $\hat{y}_a^+|0\rangle = |1,a\rangle$, $\hat{y}_a^-|1,b\rangle = \delta_{ab}|0\rangle$, $\hat{y}_a^+|1,b\rangle = c_{ab}|2\rangle$, and $\hat{y}_a^-|2\rangle = \sum_b \lambda_{ab}|1,b\rangle$; and the non-zero matrix elements of $\hat{x}_a^\pm$ are $\hat{x}_a^+|0\rangle = |1,a\rangle$, $\hat{x}_a^-|1,b\rangle = \delta_{ab}|0\rangle$, $\hat{x}_a^+|1,b\rangle = c_{ba}|2\rangle$, and $\hat{x}_a^-|2\rangle = \sum_b \lambda_{ba}|1,b\rangle$. The action of the operators $x_i^\pm, \hat{y}_i^\pm$ and n on the orthonormal basis states are shown in Fig. S1.

The spin model Hamiltonian in Eq. (17) is not Hermitian for the R-matrix in Ex. 4 for $m \geq 3$ [7], since $\hat{x}_a^+ \neq (\hat{x}_a^-)^\dagger, \hat{y}_a^+ \neq (\hat{y}_a^-)^\dagger$. However, $\hat{H}$ is parity-time symmetric [8, 9], and therefore has real eigenvalues and generates unitary time evolution. To be precise, let P be the parity operator that generates the chain reflection symmetry, and let T be the time-reversal symmetry, which, in our spin model, is simply complex conjugation. Using the explicit representation of the matrices λ_{ab}, c_{ab} in Tab. I and footnote [10], we see that $\lambda_{ab}^* = \lambda_{ba}, c_{ab}^* = c_{ba}$, and therefore, the time-reversal operation T simply swaps the operators $\hat{x}_a^\pm \leftrightarrow \hat{y}_a^\pm$ in the Hamiltonian $\hat{H}$, which can subsequently be undone by P. Thus, $\hat{H}$ is invariant under the combined operation PT, and all eigenvalues of $\hat{H}$ are real (as already seen from the exact solution of the spectrum).

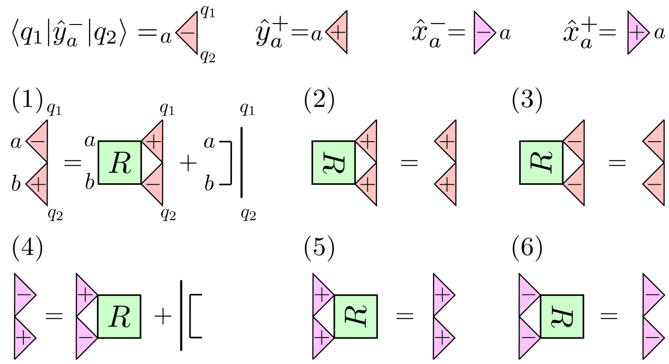

FIG. S2. Graphical representation of the CRs between the local spin operators $\{\hat{x}_a^\pm, \hat{y}_a^\pm\}_{a=1}^m$ in Eq. (S12). The matrix elements of each operator $\{\hat{x}_a^\pm, \hat{y}_a^\pm\}_{a=1}^m$ is a tensor (represented by the triangles) with two quantum indices (e.g. the indices q_1 and q_2 shown in figure) and one auxiliary index (e.g. the index a), and the R-matrix (represented by a square) is a tensor with four auxiliary indices. Matrix multiplication goes from top to bottom in the quantum space and from left to right in the auxiliary space.

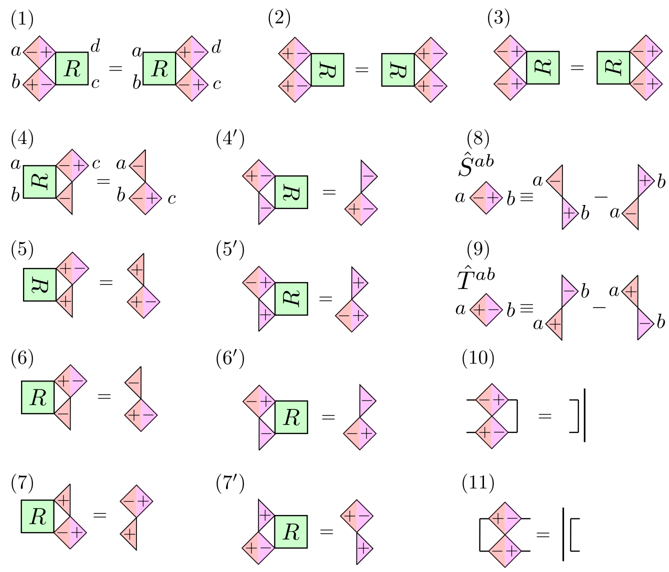

FIG. S3. Graphical representation of the CRs between the local spin operators $\hat{S}^{ab}, \hat{T}^{ab}$ and $\{\hat{x}_a^\pm, \hat{y}_a^\pm\}_{a=1}^m$.

B. Generalized Jordan-Wigner transformations

We now prove that the emergent paraparticles creation and annihilation operators defined by the MPO JWT in Eq. (18) do satisfy the parastatistical CRs in Eq. (6), and the spin Hamiltonian in Eq. (17) is mapped to the free paraparticle Hamiltonian in Eq. (19). An important first step is to prove the algebraic relations between the local spin operators $\hat{S}^{ab}, \hat{T}^{ab}$ and $\{\hat{x}_a^\pm, \hat{y}_a^\pm\}_{a=1}^m$ as shown graphically in Fig. S3. For the R-matrices given in Tab. I, all these relations can be explicitly checked by hand, but to prove them for an arbitrary R-matrix, we need to use some techniques of representation the-

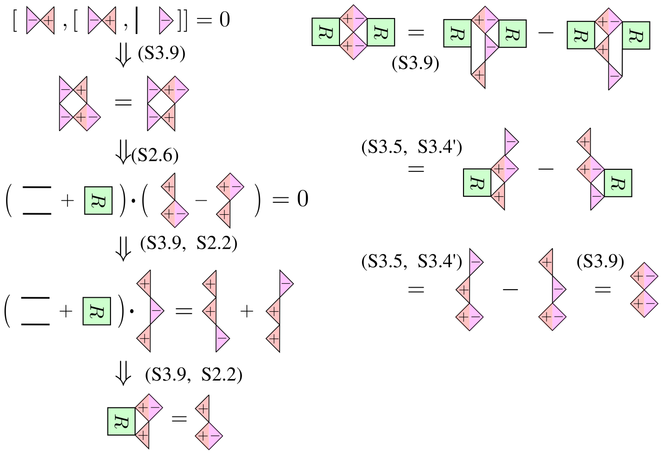

FIG. S4. The graphical proof of relations (S2.2) (right panel) and (S2.5) (left panel) in Fig. S3.

ory. Here we show the proof of relations (2) and (5) as examples, while others can be proven in a similar way. First, it is straightforward to verify that the operators $\hat{S}^+ = \sum_a \hat{x}^+_{1a}\hat{y}^-_{2a}, \hat{S}^- = \sum_a \hat{x}^-_{1a}\hat{y}^+_{2a}, \hat{S}^z = (\hat{n}_1 - \hat{n}_2)/2$ form a representation of the $\mathfrak{sl}_2$ Lie algebra. This can be checked by direct computation using Eqs. (S12,S13). In addition, we have $[\hat{S}^+, \hat{x}^-_{2a}] = 0$ and $[\hat{S}^z, \hat{x}^-_{2a}] = \hat{x}^-_{2a}/2$, so under the adjoint action of this $\mathfrak{sl}_2$, $\hat{x}^-_{2a}$ is a highest weight state with $S^z = 1/2$. The only finite dimensional representation of $\mathfrak{sl}_2$ corresponding to this highest weight state is the spin-1/2 representation, in which we have $[\hat{S}^-, [\hat{S}^-, \hat{x}^-_{2a}]] = 0$. Relation (5) in Fig. S3 can be proven by expanding this equation and using the fifth relation in Eq. (S12), as shown in the left panel of Fig. S4. Then relation (2) can be proven from relation (5) as shown in the right panel of Fig. S4, using tensor graphical manipulations.

With all those relations shown in Fig. S3, Eq. (6) can be proven. For example, in Fig. S5 we show the proof of the first parastatistical commutation relation in Eq. (6) for the $\hat{\psi}^{\pm}_{i,a}$ defined in terms of the spin operators in Eq. (18) and the algebraic relations in Fig. S3. Other relations in Eq. (6) are proven in a similar way. Furthermore, one can insert Eq. (18) into Eq. (19) to reproduce Eq. (17), using the last two relations in Fig. S3 and a similar graphical manipulation as in Fig. S5. This proves the exact mapping from the 1D spin model to free paraparticles.

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[6] Here we are using $\mathcal{X}_R$ (generated by $\{\hat{\psi}^{\pm}_a\}^m_{a=1}$) and $\mathcal{X}^+_R$ (generated by $\{\hat{\psi}^+_a\}^m_{a=1}$) as auxiliary tools to construct the local Hilbert space. One should not confuse the $\hat{\psi}^{\pm}_a$ here (which is defined only on the local Hilbert space) with the paraparticle creation/annihilation operators we obtain from the MPO JWT in Eq. (18), which are non-local operators acting on the global Hilbert space.

[7] The model is still well-defined for $m = 2$, where $\hat{H}$ is Hermitian. But that case is trivial: when $m = 2$, $\hat{H}$ is equal to the sum of two decoupled chains of XY models.

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[10] For example, we can take $\lambda = e^{-M}, c = -e^{M}$, where M is an $m \times m$ antisymmetric matrix, $M^T = -M$, with complex entries satisfying $\mathrm{Tr}[e^{-2M}] = -2$ (one can get an explicit solution using a block-diagonal ansatz for M with maximum block size 2).

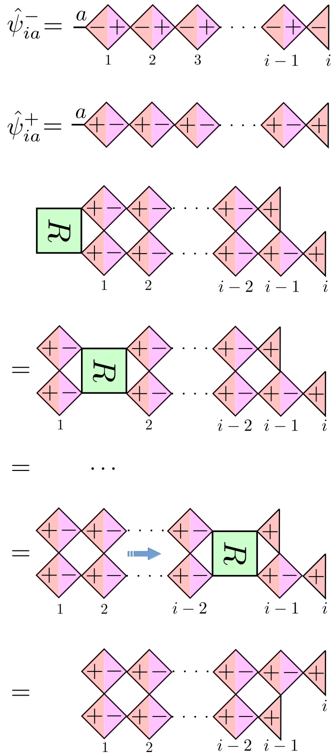

FIG. S5. The graphical proof of the first relation in Eq. (6), using the definition of $\hat{\psi}^{\pm}_{i,a}$ in Eq. (18) and the algebraic relations in Fig. S3.