

Sub-dimensional magnetic polarons in the one-hole doped SU(3) t -J model

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The physics of doped Mott insulators is at the heart of strongly correlated materials and is believed to constitute an essential ingredient for high-temperature superconductivity. In systems with higher SU(N) spin symmetries, even richer magnetic ground states appear at a filling of one particle per site compared to the case of SU(2) spins, but their fate upon doping remains largely unexplored. Here we address this question by studying a single hole in the SU(3) t -J model, whose undoped ground state features long-range, diagonal spin stripes. By analyzing both ground state and dynamical properties utilizing the density matrix renormalization group, we establish the appearance of magnetic polarons consisting of chargons and flavor defects, whose dynamics is constrained to a single effective dimension along the ordered diagonal. We semi-analytically describe the system using geometric string theory, where paths of hole motion are the fundamental degrees of freedom. With recent advances in the realization and control of SU(N) Fermi-Hubbard models with ultracold atoms in optical lattices, our results can directly be observed in quantum gas microscopes with single-site resolution. Our work suggests the appearance of intricate ground states at finite doping constituted by emergent, coupled Luttinger liquids along diagonals, and is a first step towards exploring a wealth of physics in doped SU(N) Fermi-Hubbard models on various geometries.

Introduction.— The physics of doped Mott insulators, dominated by the intricate competition between kinetic and interaction energies, is at the heart of strongly correlated physics. For two-flavor spins, the two-dimensional (2D) SU(2) symmetric Fermi-Hubbard (FH) model represents a paradigmatic model believed to capture some essential physics of high-temperature superconducting cuprates [1, 2]. Nevertheless, though tremendous progress has been made both numerically [3–9] and experimentally [10–18], the microscopic mechanisms behind many of the phases appearing in the FH model remain to be fully understood.

Higher spin symmetries, realized by SU($N > 2$) invariant models, promise rich physics beyond the SU(2) paradigm [19–28]. In addition to novel magnetic states and enhanced quantum fluctuations, they separate features that are linked in the vanilla SU(2) Hubbard model: perfect nesting and van Hove singularities at one particle per site are absent, in contrast to the SU(2) FH model. At unit filling $\langle \hat{n} \rangle = 1$, the SU(3) FH model has been shown to feature rich magnetic structures of various translation symmetry breaking patterns, where in particular finite repulsive interactions $U > 0$ are necessary to open a charge gap and observe magnetically ordered Mott insulating states [23, 24, 29–31]. In the strongly coupled regime, effective mappings to SU(3) Heisenberg models at $\langle \hat{n} \rangle = 1$ reveal a 3-sublattice (3-SL), diagonally striped magnetic order, stabilized by quantum fluctuations through an order by disorder mechanism [32, 33].

An important setting for the SU(N) FH model is ultracold alkaline-earth-atoms (AEAs), where a highly precise SU(N) symmetry arises from the near-perfect decoupling

of nuclear spins from the electronic structure due to their closed shells [34–38]. Using fermionic isotopes of ⁸⁷Sr and ¹⁷³Yb, recent ultracold atom experiments have successfully observed Mott insulating states [39–41], nearest-neighbor (NN) antiferromagnetic correlations [42, 43], as well as measured the equation of state in the SU(6) FH model [44].

While the intricate magnetic structures of SU(N) Heisenberg magnets have been studied with increased interest [45–49], the physics of doped SU(N) Mott insulators remains widely unexplored. Studying doped Mott insulators with higher spin symmetries promises novel insights into the competition between spin and motional degrees of freedom, possibly helping to unravel the microscopic nature of hole pairing in strongly correlated electronic systems.

In this letter, we utilize tensor network methods to study the singly doped SU(3) t -J model, both in and out of equilibrium. We establish the appearance of magnetic polarons consisting of chargons and flavor defects (SU(3) spinons), whose dynamics is constrained to a single effective dimension along the ordered diagonal. We demonstrate how this sub-dimensional phenomenology is qualitatively captured within non-linear geometric string theory after including both chargon and spinon fluctuations of the magnetic polaron. Our results can be directly probed with ultracold AEAs paired with quantum gas microscopes with single-site resolution [50–52], paving the way towards exploring a wealth of exotic physics in doped SU(N) symmetric systems.

The model.— In the limit of strong on-site repulsion, the hole-doped 2D SU(3) symmetric FH model as realized

by AEA in sufficiently deep lattices at density $\langle n \rangle \leq 1$ reduces to the SU(3) symmetric t - J model on the square lattice. Neglecting three-site terms, the corresponding Hamiltonian reads

$$\hat{\mathcal{H}} = -t \sum_{\langle \mathbf{i}, \mathbf{j} \rangle, \alpha} \hat{\mathcal{P}} \left(\hat{c}_{\mathbf{i}, \alpha}^\dagger \hat{c}_{\mathbf{j}, \alpha} + \text{h.c.} \right) \hat{\mathcal{P}} + \frac{J}{2} \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} \left(\sum_{\alpha, \beta} \hat{\mathcal{P}} \hat{c}_{\mathbf{i}, \alpha}^\dagger \hat{c}_{\mathbf{i}, \beta} \hat{c}_{\mathbf{j}, \beta}^\dagger \hat{c}_{\mathbf{j}, \alpha} \hat{\mathcal{P}} - \hat{n}_{\mathbf{i}} \hat{n}_{\mathbf{j}} \right). \quad (1)$$

Here, $\hat{c}_{\mathbf{i}, \alpha}^\dagger$ ($\hat{c}_{\mathbf{i}, \alpha}$) are creation (annihilation) operators of flavor $\alpha = \{\text{R, G, B}\}$ on site \mathbf{i} , $\langle \mathbf{i}, \mathbf{j} \rangle$ denotes nearest neighbors on the 2D square lattice, $\hat{n}_{\mathbf{i}}$ are the local particle densities, and $\hat{\mathcal{P}}$ is the Gutzwiller operator projecting out states with local occupancy > 1 . Note that for $N = 2$, the second part of Eq. (1) reduces to $J \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} (\hat{\mathbf{S}}_{\mathbf{i}} \cdot \hat{\mathbf{S}}_{\mathbf{j}} - \hat{n}_{\mathbf{i}} \hat{n}_{\mathbf{j}} / 4)$ as in the usual t - J model, with $\hat{\mathbf{S}}_{\mathbf{i}}$ the spin-1/2 operators [53]. For a single hole, the density-density interaction in Eq. (1) merely leads to a constant energy shift (up to boundary effects) and is neglected in the following [54].

Ground state.— We simulate the ground state of the one-hole doped SU(3) t - J model, Eq. (1), using the density matrix renormalization group (DMRG) [55–59]. We implement separate U(1) particle conservation symmetries for each flavor, and simulate systems of size $L_x \times L_y = 9 \times 5$. Using bond dimensions of $\chi = 5000$, we carefully ensure convergence of all observables. In the undoped case, open boundary conditions (OBC) have been identified as crucial to observe a diagonally striped ground state in numerically available systems sizes and were argued to capture the physics of the thermodynamic limit more accurately than periodic boundaries [33, 54]. Therefore, also in the case of the hole-doped SU(3) t - J model, we focus on OBC along both x - and y - directions in the following.

By introducing local chemical potentials $-\mu_p \sum_{i \in \text{edge}} \hat{n}_{\alpha, i}$ at the short edges of the system, we explicitly break the SU(3) symmetry and pin a 3-sublattice (3-SL) stripe order along the diagonal, indicated by colored lines in Fig. 1 (a). We introduce a single hole into the system by removing a particle corresponding to the flavor of the central site \mathbf{i}_0 in a (classically) ordered background. For the pinning shown in Fig. 1, this corresponds to the symmetry sector $N_G = N_B = L_x L_y / 3$, $N_R = L_x L_y / 3 - 1$, where N_α is the total particle number of flavor $\alpha = \text{R}$ (red), G (green), B (blue).

The hole density distribution of the ground state determined by DMRG, $\langle \hat{n}_{\mathbf{i}}^h \rangle = 1 - \sum_\alpha \langle \hat{n}_{\mathbf{i}}^\alpha \rangle$, is shown in Fig. 1 (a) for $t/J = 1.5$. The hole density features a pronounced anisotropy, whereby its distribution has enhanced weight on the diagonals aligning with the pinned order. On the other hand, the hole density is suppressed

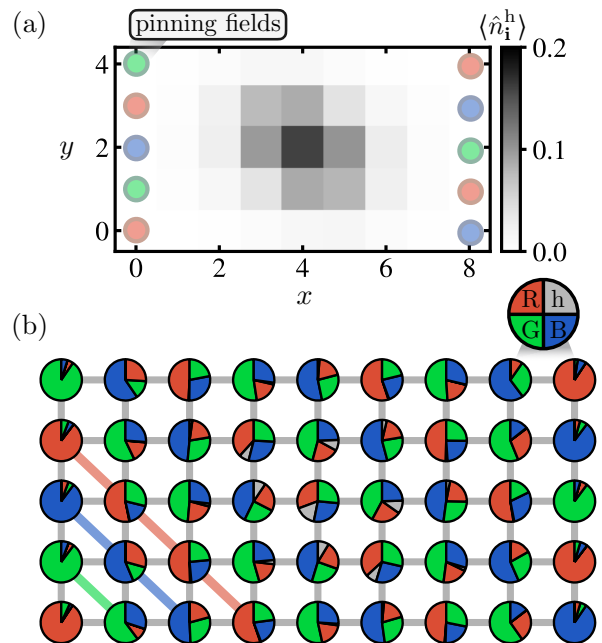


FIG. 1. **Single hole ground state.** (a) DMRG results of the hole density distribution $\langle \hat{n}_{\mathbf{i}}^h \rangle$ for a single hole (missing red atom) doped into a magnetic background pinned at the edges (indicated by the colored circles along the edges), for a $L_x \times L_y = 9 \times 5$ system with open boundaries, $t/J = 1.5$, and pinning fields $\mu_p/J = 1$. The hole distribution is anisotropic, such that the hole spreads further along the diagonal spin stripes than perpendicular to them. The full on-site moment distributions $\langle \hat{n}_{\mathbf{i}}^f \rangle$, $f = \text{h, R, G, B}$, are shown in gray, red, green and blue, respectively, in (b). The 3-SL diagonal stripe order pinned by the local chemical potentials on the boundaries is indicated by the solid colored lines in (b).

on the anti-diagonals, i.e. directions that are perpendicular to the 3-SL order. This is corroborated in Fig. 1 (b), where the full on-site moments $\langle \hat{n}_{\mathbf{i}}^f \rangle$, for $f = \text{h}$ (hole), R (red), G (green) and B (blue) are shown. Importantly, the delocalization of the hole only slightly disturbs the magnetic background in its vicinity, leaving the overall 3-SL order intact, as illustrated by the solid colorful lines in the lower left corner of Fig. 1 (b).

Geometric string theory.— In the following, we describe the doped hole in the 3-SL background using nonlinear geometric string theory (NLST) [60, 61] and establish the formation of a sub-dimensional magnetic polaron whose motion is predominantly aligned with the ordered background. The starting point is a parton representation of the SU(3) t - J model, where the creation and annihilation operators are decomposed into bosonic chargons ($\hat{h}_{\mathbf{i}}$) and fermionic spinons ($\hat{f}_{\mathbf{i}, \alpha}$), $\hat{c}_{\mathbf{i}, \alpha} = \hat{h}_{\mathbf{i}}^\dagger \hat{f}_{\mathbf{i}, \alpha}$. The spinon label α corresponds to the flavor of the particle that has been removed. The single occupancy constraint in the t - J model is ensured via $\sum_\alpha \hat{f}_{\alpha, i}^\dagger \hat{f}_{\alpha, i} + \hat{h}_{\mathbf{i}}^\dagger \hat{h}_{\mathbf{i}} = 1$ for all \mathbf{i} .

We describe the magnetic polaron within the geomet-

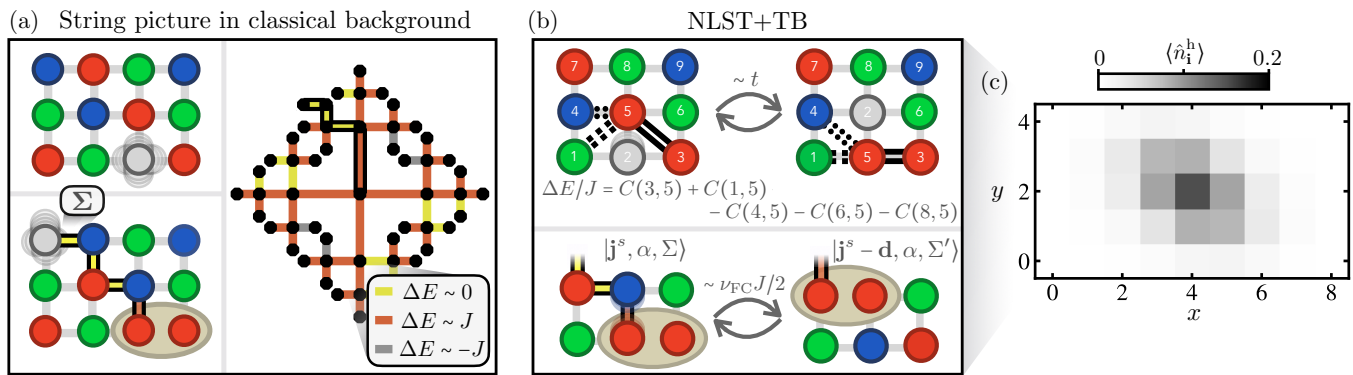


FIG. 2. **Non-linear string theory.** (a) Illustration of hole motion through a classically ordered SU(3) background. When hopping, the hole leaves behind a string Σ of displaced spins. The first hop is confining due to positive spin-spin correlations along the diagonal – leading to the formation of a spinon, see the other ellipse in the lower left panel. Subsequent hops along the diagonals, however, lead to classically degenerate configurations and paths of no additional energy cost (yellow). The full energy landscape is illustrated on the Bethe lattice in the right panel. (b) We describe chargon fluctuations within the FSA. String state energies are determined by a sum of particle exchange correlations $C(\mathbf{i}, \mathbf{j})$ of the undoped ground state, where \mathbf{i}, \mathbf{j} become neighbors after the hole is displaced through the string (upper panel). Spinon fluctuations are included by considering dominant off-diagonal couplings (lower panel), leading to an effective 1D spinon Hamiltonian on the diagonal. (c) The resulting hole density distribution when including both chargon and spinon fluctuations, matching DMRG results presented in Fig. 1.

ric string basis: By doping a single hole at position \mathbf{j}^s into the ground state $|\Psi_0\rangle$ of the undoped SU(3) Heisenberg Hamiltonian, we define the state $|\mathbf{j}^s, \alpha, \Sigma = 0\rangle = \hat{c}_{\alpha, \mathbf{j}^s} |\Psi_0\rangle$. To describe the partons, we work in the regime $t/J \gg 1$, where fluctuations of the chargon and the ordered background approximately decouple. In a first step, we fix the initial hole position \mathbf{j}^s , and describe the fast chargon fluctuations on time scales $\sim 1/t$. Motivated by the separation of energy scales, we work in the frozen spin approximation (FSA): When the hole fluctuates, the background spins are displaced and their positions change, however their quantum state is assumed to remain unaffected. This generalizes the notion of squeezed space [62–64] to two dimensions. By displacing particles in real space, the hole motion changes the underlying geometry of the lattice in squeezed space, whereby nearest neighbor (NN) pairs in squeezed space can become next-nearest neighbor (NNN) or even larger-distance pairs. String states are defined by $|\mathbf{j}^s, \alpha, \Sigma\rangle = \hat{G}_\Sigma |\mathbf{j}^s, \alpha, \Sigma = 0\rangle$, where the string operator $\hat{G}_\Sigma = \prod_{(\mathbf{i}, \mathbf{j}) \in \Sigma} (\hat{h}_i^\dagger \hat{h}_j \sum_\alpha \hat{f}_{\mathbf{j}, \alpha}^\dagger \hat{f}_{\mathbf{i}, \alpha})$ displaces the background spins along string Σ .

The string states and the tunnel coupling between them can be mapped to a Bethe lattice, illustrated in Fig. 2 (a) for a classical spin background. In particular, each string of displaced particles Σ can be associated with a corresponding potential energy on the Bethe lattice, $E(\mathbf{j}^s, \Sigma)$ [65]. Alignment of flavors along the diagonals surrounding the hole leads to an energy penalty $\Delta E \sim J$ when the hole leaves its initial position and moves by one lattice site, leaving behind a flavor defect (spinon) at \mathbf{j}^s [see the other ellipse in the lower left panel of Fig. 2 (a)]. After its initial hop, however, the en-

ergy landscape for subsequent chargon motion loses its isotropy. In particular, paths along the diagonally ordered background merely lead to local flavor exchanges along the string, which is classically degenerate with the initial 3-SL order. This, in turn, leads to the existence of string segments of no additional energy cost, illustrated by yellow paths on the Bethe lattice in the right panel of Fig. 2 (a). Directions perpendicular to the diagonal stripe order, in contrast, are associated with linearly growing magnetic energy (linear confinement), shown by red paths in Fig. 2 (a). Negative energy differences (gray lines) result from loop effects.

In the case of a classical background, string states $|\mathbf{j}^s, \alpha, \Sigma\rangle$ are mutually orthonormal except for special loop configurations which restore the ordered background, known as Trugman loops [60, 61, 66]. In the case of a (classical) diagonally striped background, these configurations involve at least 12 string segments (corresponding to three loops around a square plaquette), and are hence negligible compared to the exponential number of string states. Due to strong diagonal-order correlations in the undoped SU(3) t - J model [54], we expect that the approximation of mutual orthonormality of string states remains accurate away from the classical limit.

Our first step to go beyond the classically ordered magnetic background is to use the FSA: Upon creation of the hole, the background spins are labeled according to their original positions. When the hole moves, the particles are displaced and their positions change, resulting in energies $E(\mathbf{j}^s, \Sigma)$ of string states $|\mathbf{j}^s, \alpha, \Sigma\rangle$ that depend on correlations $C(\mathbf{i}, \mathbf{j}) = \sum_{\alpha, \beta} \langle \Psi_0 | \hat{\mathcal{P}} \hat{c}_{\alpha, \mathbf{i}}^\dagger \hat{c}_{\beta, \mathbf{i}} \hat{c}_{\beta, \mathbf{j}}^\dagger \hat{c}_{\alpha, \mathbf{j}} \hat{\mathcal{P}} | \Psi_0 \rangle / 2$ of the undoped ground state; see the upper panel of Fig. 2 (b) for an explicit illustration for $|\Sigma| = 1$. The

effective Hamiltonian reads

$$\hat{\mathcal{H}}_{\text{charge}}(\mathbf{j}^s, \alpha) = -t \sum_{\langle \Sigma, \Sigma' \rangle} |\mathbf{j}^s, \alpha, \Sigma\rangle \langle \mathbf{j}^s, \alpha, \Sigma'| + \text{h.c.} \\ + \sum_{\Sigma} E(\mathbf{j}^s, \Sigma) |\mathbf{j}^s, \alpha, \Sigma\rangle \langle \mathbf{j}^s, \alpha, \Sigma|, \quad (2)$$

with $\langle \Sigma, \Sigma' \rangle$ denoting two neighboring sites on the Bethe lattice. The ground state of the chargon Hamiltonian, Eq. (2), is given by $|\Psi_{\text{charge}}(\mathbf{j}^s, \alpha)\rangle = \sum_{\Sigma} \psi_{\mathbf{j}^s, \Sigma}^{\text{charge}} |\mathbf{j}^s, \alpha, \Sigma\rangle$, with corresponding energy $E_{\text{charge}}^0(\mathbf{j}^s)$. Though the Bethe lattice within NLST features anisotropic confining potentials, we find that the mere kinetics of the fast chargon does not account for the strong diagonal alignment of the hole density seen in DMRG simulations; hole delocalization around the spinon is almost isotropic, forming a bound magnetic polaron [54].

Instead, it is the effect of spinon fluctuations on time scales $\propto 1/J$ that lead to the observed alignment along the diagonal in Fig. 1 (a), which we include on top of chargon fluctuations by using a tight-binding description of spinon motion (NLST+TB). Concretely, we consider off-diagonal couplings $J_s(\mathbf{j}^s, \mathbf{j}^{s'}; \Sigma, \Sigma') = \langle \mathbf{j}^{s'}, \alpha, \Sigma' | \hat{\mathcal{H}} | \mathbf{j}^s, \alpha, \Sigma \rangle$ within the geometric string basis construction to describe spinon fluctuations. Owing to the Hamiltonian's $U(1)^{\otimes N}$ particle conservation symmetry, applying $\hat{\mathcal{H}}$ to string states $|\mathbf{j}^s, \alpha, \Sigma\rangle$ conserves the flavor of the removed particle α . This results in a dominantly diagonal NNN hopping of the spinon, illustrated in the lower panel of Fig. 2 (b). Here, a particle exchange of two spins in a given string state results in configurations $|\mathbf{j}^s \pm \mathbf{d}, \alpha, \Sigma'\rangle$, with $\mathbf{d} = \mathbf{e}_x - \mathbf{e}_y$ pointing along the diagonal and $|\Sigma'| = |\Sigma| - 2$. Due to finite overlaps between string states with different initial hole positions \mathbf{j}^s , the spinon hopping is reduced by a Franck-Condon factor ν_{FC} , leading to an effective diagonal hopping of $J^* \approx \nu_{\text{FC}} J/2$ [54, 61], where $\nu_{\text{FC}} = 1/2$ in the limit $t/J \gg 1$.

More formally, for each initial hole position \mathbf{j}^s we make a strong coupling Born-Oppenheimer-type product ansatz $|\Psi(\mathbf{j}^s, \alpha)\rangle = |\Psi_{\text{charge}}(\mathbf{j}^s, \alpha)\rangle \otimes |\Psi_{\text{spinon}}(\mathbf{j}^s, \alpha)\rangle \otimes |\Psi_0\rangle_{\text{sq}}$ describing the chargon, spinon and background state in squeezed space, respectively. Our trial state is then given by a superposition of composite spinon-charge states for varying \mathbf{j}^s ,

$$|\Psi\rangle = \sum_{\mathbf{j}^s \in \mathcal{D}} \psi_{\mathbf{j}^s}^{\text{spinon}} |\Psi(\mathbf{j}^s, \alpha)\rangle \\ = \sum_{\mathbf{j}^s \in \mathcal{D}} \psi_{\mathbf{j}^s}^{\text{spinon}} \sum_{\Sigma} \psi_{\mathbf{j}^s, \Sigma}^{\text{charge}} |\mathbf{j}^s, \alpha, \Sigma\rangle. \quad (3)$$

We make an additional approximation and determine $\psi_{\mathbf{j}^s}^{\text{spinon}}$ from an effective 1D single particle tight-binding description on the central diagonal (\mathcal{D}) with hopping J^* and on-site energies $E_{\text{charge}}^0(\mathbf{j}^s)$ [54]. The density in the

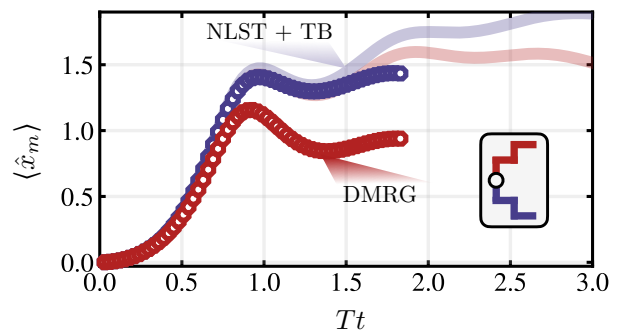


FIG. 3. **Hole dynamics.** Time evolution of an initially localized hole at $\mathbf{i}_0 = [4, 2]$. Mean Manhattan distances along the diagonal (anti-diagonal) are shown in blue (red) for a $L_x \times L_y = 9 \times 5$ system. Light red and blue lines show NLST+TB results for an infinite system with $\nu_{\text{FC}} = 0.5$.

ground state is determined by mapping $|\psi_{\mathbf{j}^s}^{\text{spinon}} \psi_{\mathbf{j}^s, \Sigma}^{\text{charge}}|^2$ back to the original real space lattice, shown in Fig. 2 (c). Note that we treat ν_{FC} as an effective free parameter of the theory, matching DMRG results strikingly well for $\nu_{\text{FC}} = 0.1$. This agreement corroborates the validity of NLST+TB, and supports the existence of a sub-dimensional magnetic polaron in the singly doped $SU(3)$ t - J model. Our additional DMRG simulations [54] further support this picture, whereby the anisotropy in the hole density is seen to increase for rising exchange interactions J/t , while a dominating hopping $J/t \ll 1$ leads to the formation of a broad, isotropic polaron cloud. Lastly, we note that while the Hilbert space spanned by the string states grows exponentially, a systematic cutoff of chargon states far away from their initial position on the Bethe lattice allows for an efficient calculation of hole density distributions.

Dynamics. — To further study the behavior of the magnetic polaron in the $SU(3)$ t - J model, we analyze quenched hole dynamics, which is particularly accessible to ultracold atom experiments and has been probed for a single hole doped into an $SU(2)$ AFM background [67]. Specifically, we analyze the hole's dynamics after doping it in the center \mathbf{i}_0 of the undoped ground state, i.e., at time $T = 0$, the initial state is $|\Psi(T=0)\rangle = |\mathbf{i}_0, \alpha, \Sigma = 0\rangle$. We use the global subspace expansion [68] for a single time step, before switching to time dependent variational principle calculations [69]. During the time evolution, we track the Manhattan distance $\langle \hat{x}_m \rangle$ from \mathbf{i}_0 along the diagonal and anti-diagonal. At short times, fast chargon fluctuations lead to a symmetric, ballistic expansion of the hole, see Fig. 3. At times $Tt \sim 1$, a rapid slow down and apparent saturation of the hole's spread is observed, reminiscent of dynamics in the $SU(2)$ t - J model [67, 70–72]. Here, it has been established that the hole's long-time dynamics is governed by spinon dynamics, i.e., by the motion of the heavy composite magnetic polaron itself. The strong splitting between diagonal and

anti-diagonal distances appearing in the SU(3) system at times $Tt \sim 1$, shown in Fig. 3, further underlines the role of spinon delocalization in the observed anisotropy.

We describe the dynamics of the magnetic polaron within geometric string theory by again assuming a state of the form Eq. (3). As the MPS calculations are limited to small system sizes with OBC, we here focus on infinite systems (i.e. we do not consider the open boundaries when constructing the string basis) to make predictions of the dynamical formation and motion of the polaron in the thermodynamic limit. Results are shown in Fig. 3 with blue and red solid lines, where all qualitative features are in line with finite-size MPS simulations. At short times we find quantitative agreement. However, the anisotropy developing later in time is seen to be underestimated within the string theory, which is likely caused by significant finite size effects in the MPS dynamics (being particularly prominent as we are focusing on observables along the diagonal). In fact, motivated by the picture of a polaron effectively constrained to 1D, in the thermodynamic limit we expect a linear expansion (saturation) of \hat{x}_m along the diagonal (anti-diagonal) at large times, which is, however, out of reach to simulate with current methods.

Discussion. — We have studied the one-hole doped SU(3) t - J model both in- and out of equilibrium. In the ground state, we observed anisotropic hole delocalization, and established the formation of a sub-dimensional polaron by combining chargon and spinon fluctuations in an effective theory. This picture was further corroborated in calculations of the dynamics initiated from a localized hole, which can provide a direct probe of the polaron physics in SU(N) ultracold atom experiments once single-site resolution becomes available. In our setting of a doped SU(3) AFM, we have demonstrated how sub-dimensional excitations naturally emerge, reminiscent of mobility restricted fractons as appearing e.g. in three dimensional X-cube models [73, 74].

Based on our study of a single hole, we propose that SU(3) AFMs on the square lattice at finite doping are described by weakly coupled Tomonaga-Luttinger (TL) liquids of bound spinon-chargeon polarons along the diagonals,

$$\begin{aligned} \hat{H}_{\text{eff}} = & \sum_{\mathcal{D}_i} \hat{H}_{\mathcal{D}_i} + \sum_{\mathcal{D}_i \neq \mathcal{D}_j} \hat{H}_{\mathcal{D}_i, \mathcal{D}_j}^{\text{int}} \\ & + \sum_{k_n} \sum_{\mathcal{D}_i \neq \mathcal{D}_j} t_{\mathcal{D}_i, \mathcal{D}_j} \hat{c}_{k_n, \mathcal{D}_i}^\dagger \hat{c}_{k_n, \mathcal{D}_j} + \text{h.c.}, \end{aligned} \quad (4)$$

where $\hat{H}_{\mathcal{D}_i}$ is the Hamiltonian of a 1D TL liquid [75] on the i 'th diagonal \mathcal{D}_i , $\hat{H}_{\mathcal{D}_i, \mathcal{D}_j}^{\text{int}}$ is the (weak) interaction between chains \mathcal{D}_i and \mathcal{D}_j , and the last term describes particles with quasi-momentum k_n hopping between the diagonals. In particular, in the absence of inter-chain couplings in Eq. (4) we predict the appearance of power-law correlations of charges along the di-

agonal order, $\langle \hat{n}_{x+\ell, y-\ell}^h \hat{n}_{x, y}^h \rangle \propto \ell^{-\alpha}$, while correlations perpendicular to the stripe order are short-range with exponential decay, $\langle \hat{n}_{x+\ell, y+\ell}^h \hat{n}_{x, y}^h \rangle \propto e^{-\ell}$. Though inter-chain interactions are a relevant perturbation of the TL liquid, we still expect these scalings over intermediate length scales for finite couplings.

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Supplementary Materials: Sub-dimensional magnetic polarons in the one-hole doped SU(3) t -J model

1. The SU(N) t -J model

We here show that in the case of $N = 2$, Eq. (1) in the main text is equivalent to the standard t -J model without next-nearest neighbor terms,

$$\hat{\mathcal{H}} = -t \sum_{\langle \mathbf{i}, \mathbf{j} \rangle, \alpha} \hat{\mathcal{P}} \left(\hat{c}_{\alpha, \mathbf{i}}^\dagger \hat{c}_{\alpha, \mathbf{j}} + \text{h.c.} \right) \hat{\mathcal{P}} + J \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} \left(\hat{S}_i^x \hat{S}_i^x + \hat{S}_i^y \hat{S}_i^y + \hat{S}_i^z \hat{S}_i^z - \frac{1}{4} \hat{n}_i \hat{n}_j \right). \quad (\text{S1})$$

Using $\hat{S}_i^\mu = \frac{1}{2} \sum_{\alpha\alpha'} \hat{c}_{i,\alpha}^\dagger \sigma_{\alpha\alpha'}^\mu \hat{c}_{i,\alpha'}$ with σ^μ ($\mu = x, y, z$) the Pauli matrices, the second part of Eq. (S1) reads

$$\begin{aligned} & \frac{J}{4} \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} (\hat{c}_{\mathbf{i}, \uparrow}^\dagger \hat{c}_{\mathbf{i}, \downarrow} + \hat{c}_{\mathbf{i}, \downarrow}^\dagger \hat{c}_{\mathbf{i}, \uparrow}) (\hat{c}_{\mathbf{j}, \uparrow}^\dagger \hat{c}_{\mathbf{j}, \downarrow} + \hat{c}_{\mathbf{j}, \downarrow}^\dagger \hat{c}_{\mathbf{j}, \uparrow}) + (\hat{c}_{\mathbf{i}, \uparrow}^\dagger \hat{c}_{\mathbf{i}, \downarrow} - \hat{c}_{\mathbf{i}, \downarrow}^\dagger \hat{c}_{\mathbf{i}, \uparrow}) (\hat{c}_{\mathbf{j}, \downarrow}^\dagger \hat{c}_{\mathbf{j}, \uparrow} - \hat{c}_{\mathbf{j}, \uparrow}^\dagger \hat{c}_{\mathbf{j}, \downarrow}) \\ & + (\hat{c}_{\mathbf{i}, \uparrow}^\dagger \hat{c}_{\mathbf{i}, \uparrow} - \hat{c}_{\mathbf{i}, \downarrow}^\dagger \hat{c}_{\mathbf{i}, \downarrow}) (\hat{c}_{\mathbf{j}, \uparrow}^\dagger \hat{c}_{\mathbf{j}, \uparrow} - \hat{c}_{\mathbf{j}, \downarrow}^\dagger \hat{c}_{\mathbf{j}, \downarrow}) - (\hat{c}_{\mathbf{i}, \uparrow}^\dagger \hat{c}_{\mathbf{i}, \uparrow} + \hat{c}_{\mathbf{i}, \downarrow}^\dagger \hat{c}_{\mathbf{i}, \downarrow}) (\hat{c}_{\mathbf{j}, \uparrow}^\dagger \hat{c}_{\mathbf{j}, \uparrow} + \hat{c}_{\mathbf{j}, \downarrow}^\dagger \hat{c}_{\mathbf{j}, \downarrow}), \end{aligned} \quad (\text{S2})$$

which can be rewritten to

$$\begin{aligned} & \frac{J}{2} \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} \hat{c}_{\mathbf{i}, \uparrow}^\dagger \hat{c}_{\mathbf{i}, \downarrow} \hat{c}_{\mathbf{j}, \downarrow}^\dagger \hat{c}_{\mathbf{j}, \uparrow} + \hat{c}_{\mathbf{i}, \downarrow}^\dagger \hat{c}_{\mathbf{i}, \uparrow} \hat{c}_{\mathbf{j}, \uparrow}^\dagger \hat{c}_{\mathbf{j}, \downarrow} + \hat{c}_{\mathbf{i}, \uparrow}^\dagger \hat{c}_{\mathbf{i}, \uparrow} \hat{c}_{\mathbf{j}, \uparrow}^\dagger \hat{c}_{\mathbf{j}, \uparrow} + \hat{c}_{\mathbf{i}, \downarrow}^\dagger \hat{c}_{\mathbf{i}, \downarrow} \hat{c}_{\mathbf{j}, \downarrow}^\dagger \hat{c}_{\mathbf{j}, \downarrow} - (\hat{n}_{\mathbf{i}, \uparrow} \hat{n}_{\mathbf{j}, \downarrow} + \hat{n}_{\mathbf{i}, \downarrow} \hat{n}_{\mathbf{j}, \uparrow} + \hat{n}_{\mathbf{i}, \uparrow} \hat{n}_{\mathbf{j}, \uparrow} + \hat{n}_{\mathbf{i}, \downarrow} \hat{n}_{\mathbf{j}, \downarrow}) \\ & = \frac{J}{2} \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} \left(\sum_{\alpha, \beta} \hat{c}_{\alpha, \mathbf{i}}^\dagger \hat{c}_{\beta, \mathbf{i}} \hat{c}_{\beta, \mathbf{j}}^\dagger \hat{c}_{\alpha, \mathbf{j}} - \hat{n}_i \hat{n}_j \right). \end{aligned} \quad (\text{S3})$$

This corresponds to Eq. (1) in the main text. We note that in our calculations, we only implement the hopping and exchange term, i.e., the density-density interaction $-J/2 \sum_{\langle \mathbf{i}, \mathbf{j} \rangle} \hat{n}_i \hat{n}_j$ is not considered. In the case of a single dopant, this only constitutes a constant energy shift in the Hamiltonian in the thermodynamic limit.

2. Geometric string theory

In this section, we give a more detailed explanation of the non-linear geometric string theory for the doped SU(3) t -J model. As described in the main text, we describe chargon fluctuations within the frozen spin approximation (FSA) in a first step, before adding spinon motion on top in a tight-binding manner.

Chargon fluctuations. The crucial ingredient in order to describe fluctuations of the chargon around its initial position \mathbf{j}^s within NLST are magnetic correlations of the undoped system,

$$C(\mathbf{i}, \mathbf{j}) = \sum_{\alpha, \beta} \langle \Psi_0 | \hat{\mathcal{P}} \hat{c}_{\alpha, \mathbf{i}}^\dagger \hat{c}_{\beta, \mathbf{i}} \hat{c}_{\beta, \mathbf{j}}^\dagger \hat{c}_{\alpha, \mathbf{j}} \hat{\mathcal{P}} | \Psi_0 \rangle / 2, \quad (\text{S4})$$

where $|\Psi_0\rangle$ is the ground state with one particle per site. Fig. S1 shows correlations for a fixed reference site $\mathbf{i} = [x = 4, y = 2]$ in the center of the finite-size system we study by DMRG. Following the 3-SL order of the ground state, sites are correlated positively along every third diagonal; nearest neighbor correlations, in contrast, show strong negative signals. When a hole is added at initial position \mathbf{j}^s and then hops away, assuming a frozen spin background it reshuffles the particles in its vicinity, leading to an energy cost that can directly be evaluated from correlations given by Eq. (S4).

For instance, if a hole initially placed in the center $\mathbf{j}^s = [4, 2]$ moves up by one lattice site, the corresponding change in magnetic energy is given by $\Delta E/J = C(\mathbf{i} = [3, 2], \mathbf{j} = [4, 3]) + C(\mathbf{i} = [5, 2], \mathbf{j} = [4, 3]) + C(\mathbf{i} = [4, 1], \mathbf{j} = [4, 3]) - C(\mathbf{i} = [4, 4], \mathbf{j} = [4, 3]) - C(\mathbf{i} = [5, 3], \mathbf{j} = [4, 3]) - C(\mathbf{i} = [3, 3], \mathbf{j} = [4, 3]) = 0.97$. Using correlations Eq. (S4), all energies on the Bethe lattice are calculated this way, depicted in Fig. S1 (b) for $\mathbf{j}^s = [4, 2]$. Note that this includes the effect of open boundary conditions within NLST. Additionally, if the hole moves outside the finite system's frame, the site is cut off from the Bethe lattice [in Fig. S1 (b), a Bethe lattice depth of $d = 3$ is considered. The two sites corresponding to strings (up, up, up) and (down, down, down) lie outside the frame and are thus cut off]. Energies of string configurations (diagonal matrix elements of a string configuration with itself) using the system's correlations

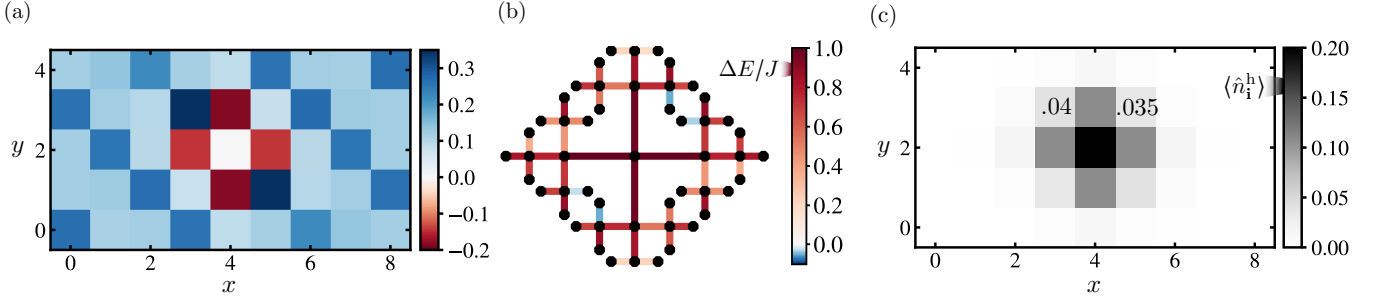


FIG. S1. **Chargon fluctuations within NLST.** (a) Correlations $C(\mathbf{i}, \mathbf{j})$, Eq. (S4), for a fixed reference site $\mathbf{i} = [4, 2]$ calculated with DMRG with the same pinning fields as described in the main text. These correlations are the essential ingredient for calculating the energy of string states on the Bethe lattice, presented for lattice depth $d = 3$ in (b). The structure follows the classical picture illustrated in Fig. 2 (a) in the main text. Finite size effects are taken into account by cutting off sites on the Bethe lattice that do not lie within the finite system's frame. (c) Density distribution in real space after diagonalizing the hopping Hamiltonian on the Bethe lattice, Eq. (S5). Though the anisotropic energy distribution on the Bethe lattice leads to slightly larger hole densities on the diagonals compared to the anti-diagonals, differences are small (see the numerical values).

follow the structure of a classical background as illustrated in Fig. 2 (a) in the main text, whereby string energies along the diagonal are lower compared to other paths.

The Hamiltonian of the chargon for an initial hole position \mathbf{j}^s is then expressed within the geometric string basis on the Bethe lattice,

$$\hat{H}_{\text{charge}}(\mathbf{j}^s, \alpha) = -t \sum_{\langle \Sigma, \Sigma' \rangle} |\mathbf{j}^s, \alpha, \Sigma\rangle \langle \mathbf{j}^s, \alpha, \Sigma'| + \text{h.c.} + \sum_{\Sigma} E(\mathbf{j}^s, \Sigma) |\mathbf{j}^s, \alpha, \Sigma\rangle \langle \mathbf{j}^s, \alpha, \Sigma|. \quad (\text{S5})$$

The Hamiltonian, Eq. (S5), is diagonalized, yielding

$$|\Psi_{\text{charge}}(\mathbf{j}^s, \alpha)\rangle = \sum_{\Sigma} \psi_{\mathbf{j}^s, \Sigma}^{\text{charge}} |\mathbf{j}^s, \alpha, \Sigma\rangle \quad (\text{S6})$$

as its ground state with eigenenergy $E_{\text{charge}}^0(\mathbf{j}^s)$. Mapping the hole density back to real space, $\langle \hat{n}_{\mathbf{j}}^h \rangle = \sum_{\Sigma \in Q_{\mathbf{j}}} |\psi_{\mathbf{j}^s, \Sigma}^{\text{charge}}|^2$ where $Q_{\mathbf{j}}$ is the set of paths leading from a hole initially at \mathbf{j}^s to be located at \mathbf{j} , results in the density distribution shown in Fig. S1 (c) for $\mathbf{j}^s = [4, 2]$. Though the anisotropic energy distribution on the Bethe lattice leads to slightly larger hole densities on the diagonals compared to the anti-diagonals, differences are small [see the numerical values in Fig. S1 (c)].

Comparing the result to the densities as acquired from DMRG calculations, this suggests that the magnetic polaron itself (built up from fast chargon fluctuations centered around the fixed spinon) is only slightly influenced by the anisotropic energies on the Bethe lattice, but instead the motion of the composite chargon-spinon object induces the observed anisotropy. This is further suggested by DMRG results for varying t/J , where an increase of t/J leads to a decrease of the density's spatial anisotropy, as shown in Fig. S2.

Tight-binding description of spinon motion. On top of chargon fluctuations, we consider the leading spinon hopping terms, i.e., we consider off-diagonal couplings $J_s(\mathbf{j}^s, \mathbf{j}^{s'}; \Sigma, \Sigma') = \langle \mathbf{j}^{s'}, \alpha, \Sigma' | \hat{H} | \mathbf{j}^s, \alpha, \Sigma \rangle$. In the case of the SU(2) symmetric t - J model, major contributions are given by next-nearest neighbor (NNN) spinon hopping processes isotropically in all spatial directions. In contrast, in the SU(3) t - J model, dominant contributions are restricted to diagonal NNN spinon hopping processes along the 3-SL order. Fig. S3 (a) illustrates the process: For a given string configuration $|\mathbf{j}^s, \alpha = \text{R}, \Sigma\rangle$, exchange of two neighboring particles (here given by the green and red flavors at the left edge of the central leg) leads to a string configuration $|\mathbf{j}^s - \mathbf{e}_x + \mathbf{e}_y, \alpha = \text{R}, \Sigma'\rangle$, with a resulting string length $|\Sigma'| = |\Sigma| + 2$ and unit vectors $\mathbf{e}_x, \mathbf{e}_y$. More generally, the Hamiltonian couples off-diagonal string states $\langle \mathbf{j}^s, \alpha, \Sigma | \hat{H} | \mathbf{j}^s \pm \mathbf{d}, \alpha, \Sigma' \rangle$ with $|\Sigma'| = |\Sigma| \pm 2$ and $\mathbf{d} = \mathbf{e}_x - \mathbf{e}_y$ pointing along the diagonal stripe order.

Owing to the finite overlap of string states with different initial hole positions \mathbf{j}^s , the effective spinon hopping is given by

$$J^*(\mathbf{j}^s, \mathbf{j}^{s'}) = \sum_{\Sigma, \Sigma'} J_s(\mathbf{j}^s, \mathbf{j}^{s'}; \Sigma, \Sigma') \psi_{\mathbf{j}^{s'}, \Sigma'}^{\text{charge}*} \psi_{\mathbf{j}^s, \Sigma}^{\text{charge}}. \quad (\text{S7})$$

In the classical limit and for large system sizes, $J_s(\mathbf{j}^s, \mathbf{j}^{s'}; \Sigma, \Sigma') = J/2$ if a single particle exchange relates the two string states $|\mathbf{j}^s, \alpha, \Sigma\rangle$ and $|\mathbf{j}^{s'}, \alpha, \Sigma'\rangle$. As the exact evaluation of $J_s(\mathbf{j}^s, \mathbf{j}^{s'}; \Sigma, \Sigma')$ is cumbersome, we approximate

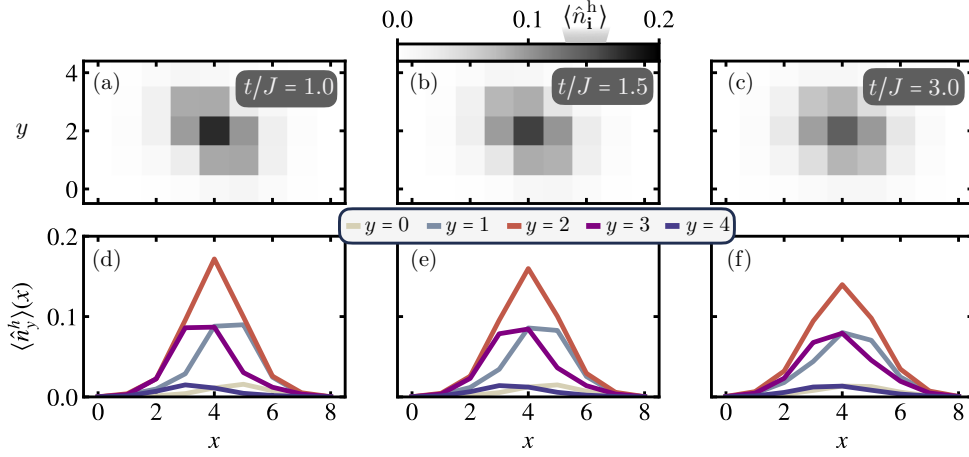


FIG. S2. **Varying t/J .** (a)-(c) 2D hole density distributions for $t/J = 1.0, 1.5, 3.0$, respectively, calculated with DMRG for $L_x \times L_y = 9 \times 5$ systems with OBC. (d)-(f) Hole densities along the x -direction, for each of the five legs. When increasing t/J (i.e. increasing the ratio of hole fluctuations compared to magnetic coupling strength), the anisotropy in the density is observed to monotonously decrease.

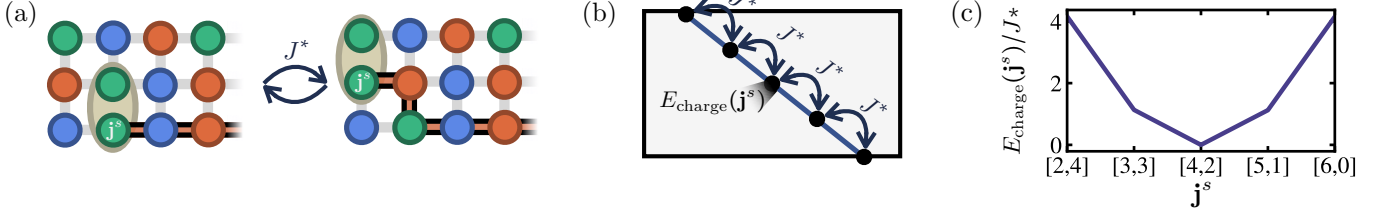


FIG. S3. **Spinon tight-binding description.** Exchange processes as appearing in the Hamiltonian of the SU(3) t - J model lead to hopping processes of the spinon, as illustrated in (a). Dominant contributions come from diagonal spinon hopping events, (b), resulting in a string state $|\Sigma', \mathbf{j}^s + \mathbf{d}\rangle$ of length $|\Sigma'| = |\Sigma| \pm 2$. The on-site energy $E_{\text{charge}}(\mathbf{j}^s)$ defines an effective potential seen by the spinon, (c), here shown for $\nu_{\text{FC}} = 0.1$.

$J^*(\mathbf{j}^s, \mathbf{j}^{s'}) \approx \nu_{\text{FC}} J/2$, where we treat the Franck-Condon factor ν_{FC} as an effective fit parameter of the geometric string theory. In the limit of weak coupling, $t \ll J$, the Franck-Condon factor approaches $\nu_{\text{FC}} = 0$. In the strong coupling regime, $t \gg J$, $\nu_{\text{FC}} \rightarrow 0.5$ [61].

We model diagonal hopping of the heavy polaron by an effectively 1D tight-binding system, with hopping parameter $J^* = \nu_{\text{FC}} J/2$ and on-site energies $E_{\text{charge}}^0(\mathbf{j}^s)$ calculated via NLST, cf. Eq. (S6), with \mathbf{j}^s lying on the diagonal \mathcal{D} that includes the central site – illustrated in Fig. S3 (b) and (c). The Hamiltonian is given by

$$\hat{\mathcal{H}}_{\text{spinon}} = J^* \left(\sum_{(i,j) \in \mathcal{D}} c_i^\dagger c_j + \text{h.c.} \right) + \sum_{\mathbf{j} \in \mathcal{D}} E_{\text{charge}}^0(\mathbf{j}) c_{\mathbf{j}}^\dagger c_{\mathbf{j}}, \quad (\text{S8})$$

yielding coefficients $\psi_{\mathbf{j}^s}^{\text{spinon}}$ for spinon positions \mathbf{j}^s . We note that, as both ν_{FC} and J are positive, the effective spinon hopping is positive, $J^* > 0$, yielding a dispersion minimum of the spinon at $k = \pi$ reminiscent to 2D quantum magnets [61, 76]. Finally, we combine charge and spinon parts by a plane-wave ansatz, arriving at

$$|\Psi\rangle = \sum_{\mathbf{j}^s \in \mathcal{D}} \psi_{\mathbf{j}^s}^{\text{spinon}} |\Psi_{\text{charge}}(\mathbf{j}^s, \alpha)\rangle = \sum_{\mathbf{j}^s \in \mathcal{D}} \psi_{\mathbf{j}^s}^{\text{spinon}} \sum_{\Sigma} \psi_{\mathbf{j}^s, \Sigma}^{\text{charge}} |\mathbf{j}^s, \alpha, \Sigma\rangle. \quad (\text{S9})$$

The total hole distribution is given by a weighed sum with coefficients $|\psi_{\mathbf{j}^s}^{\text{spinon}}|^2$ of hole distributions $|\psi_{\mathbf{j}^s, \Sigma}^{\text{charge}}|^2$ for each mean chargon position \mathbf{j}^s . Fig. S4 compares DMRG calculations with NLST including the spinon tight binding description. Indeed, for a Franck-Condon factor of $\nu_{\text{FC}} = 0.1$, the hole density is reproduced accurately for $t/J = 1.5$.

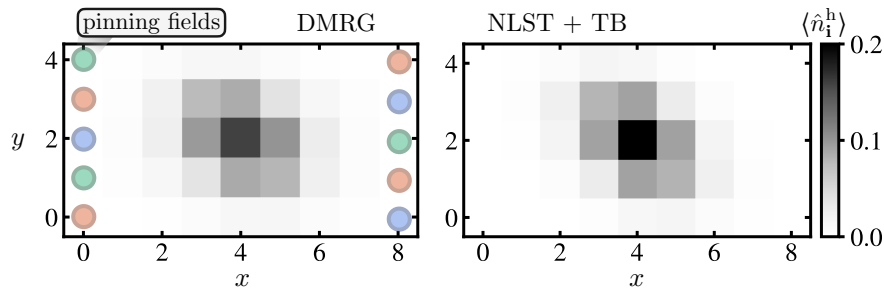


FIG. S4. **Comparison of DMRG with NLST.** Hole densities for $t/J = 1.5$ on a $L_x \times L_y = 9 \times 5$ system using DMRG (left panel), and NLST (right panel) for Bethe lattice depth $d = 6$ for $t/J = 1.5$ (a) and Franck-Condon factor $\nu_{\text{FC}} = 0.1$.

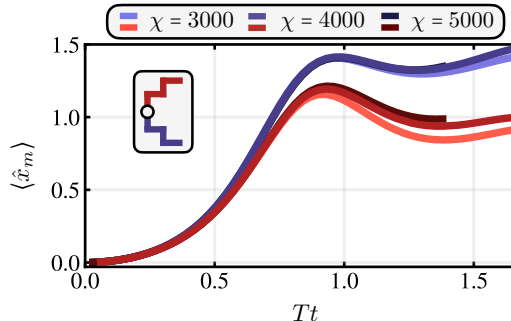


FIG. S5. **Convergence of DMRG dynamics.** Mean Manhattan distances along the diagonal and anti-diagonal in a $L_x \times L_y = 9 \times 5$ system with $t/J = 1.5$ as a function of time, for maximum bond dimensions $\chi = 3000, 4000, 5000$. While the mean Manhattan distance converges along the diagonal, slight differences between $\chi = 4000$ and $\chi = 5000$ are seen along the anti-diagonal. Nevertheless, the dynamical formation of the sub-dimensional magnetic polaron that leads to the appearance of the anisotropy at early times is identical for all bond dimensions.

3. Dynamics

We calculate the dynamical properties of a doped hole using MPS time evolution methods. As local methods suffer from large projection errors at small time steps (where the entanglement in the charge sector is zero), we use the global expansion method [68], before switching to TDVP [69] once the maximum set bond dimension is achieved. In particular, we choose a Krylov subspace order of 3, time steps $\Delta T J = 0.02$, and bond dimensions $\chi_{\text{max}} = 3000, 4000, 5000$. Fig. S5 shows the dynamics presented in Fig. 3 in the main text for the various bond dimensions. Along the diagonal, the mean Manhattan distance is seen to converge up to times $Tt \sim 1.3$. Minor deviations between bond dimensions $\chi = 4000$ and $\chi = 5000$ are visible starting from times $Tt \sim 1.0$ along the anti-diagonal. Nevertheless, the early time dynamics including the dynamical appearance of the anisotropy between the diagonal and anti-diagonal mean distances is well converged.

The geometric string theory for the chargon dynamics presented in the main text is, in analogy to the ground state considerations, based on a product state ansatz with effective Hamiltonian

$$\hat{\mathcal{H}}_{\text{eff}} = \hat{\mathcal{H}}_{\text{charge}} + \hat{\mathcal{H}}_{\text{spinon}}, \quad (\text{S10})$$

such that dynamical properties are evaluated by calculating $\exp(-i\hat{\mathcal{H}}_{\text{eff}}T) |j^s, \alpha, \Sigma = 0\rangle$; here, $\hat{\mathcal{H}}_{\text{spinon}}$ only acts on the spinon degree of freedom $|j^s\rangle$, whereas $\hat{\mathcal{H}}_{\text{charge}}$ generates chargon dynamics for a given spinon position. In order to make predictions for the formation and motion of the magnetic polaron in the thermodynamic limit, we do not account for finite size effects (as described for the ground state calculations) in the NLST+TB dynamics calculations.

4. Role of boundary conditions in the undoped SU(3) t - J model

As mentioned in the main text, we focus on open boundary conditions (OBC) in both directions in our simulations. Indeed, we find that for the accessible system sizes OBCs are crucial to observe the three sublattice diagonal stripe

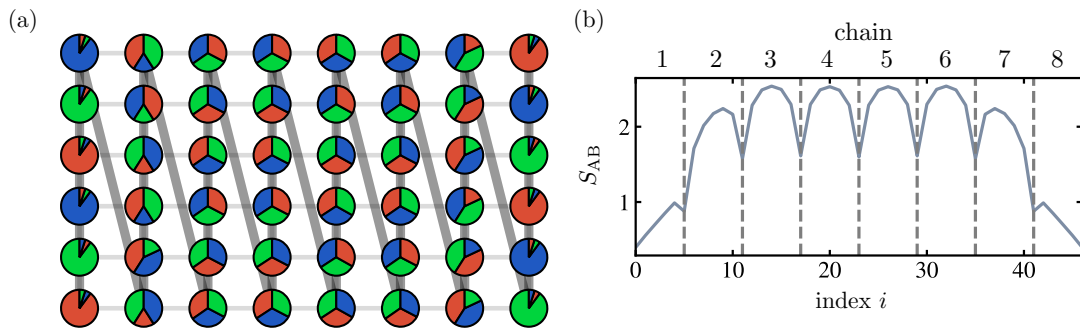


FIG. S6. **Periodic boundary conditions.** On-site moments (a) and entanglement entropy (b) for a system of size $L_x \times L_y = 8 \times 6$ with periodic boundaries along the y -direction. Both boundaries pin the 3-SL diagonal order, with pinning strength $\mu_p/J = 1$. The site index in (b) indicates the border of the subsystem, following the standard snake indexing as illustrated by dark gray lines in (a). Both entanglement entropy and on-site moments suggest the absence of magnetic order despite the applied pinning, and instead suggest the appearance of coupled 1D (periodic) chains.

order in the undoped ground state of the $SU(3)$ t - J model, consistent with what was mentioned in Ref. [33]. Fig. S6 (a) shows the on-site moments of the three flavors for a system of size $L_x \times L_y = 8, 6$ with periodic boundaries (PBC) applied along the short (y -) direction. Even with the applied pinning (shown in Fig. S6 for $\mu_p/J = 1$), the order rapidly disappears away from the boundaries. Moreover, we have carefully checked that full projections of real-space patterns do further not reveal any ordered state. For the system widths considered here, the entanglement entropy reveals that the ground state converges to a state of weakly coupled 1D (periodic) chains, as shown in Fig. S6 (b). We conclude that the appearance of weakly coupled chains in periodic systems is an artifact of finite-size effects, and that we expect diagonal stripe order to appear when $L_y \gg 1$ becomes much larger – which is, however, not accessible with current numerical techniques.